and

$$10 \vee \frac{30}{10} = \text{lcm}\{10, 3\} = 30.$$

Hence, 3 is a complement of 10 in this lattice.

REMARK 3.2.46 ▶ For any positive integer n, we can construct the lattice $(D_n, \leq)$, where D_n is the set of all positive divisors of n, $a \leq b$ if and only if a divides b. In the lattice, $a \vee b = \text{lub}\{a, b\} = \text{lcm}\{a, b\}$ and $a \wedge b = \text{glb}\{a, b\} = \text{gcd}\{a, b\}$ for any $a, b \in D_n$.

Theorem 3.2.47: In a distributive lattice $(L, \leq)$ with 1 and 0, every element has at most one complement.

Proof: Let $a \in L$. Suppose b, c are two complements of a in L. Then $a \vee b = 1$, $a \wedge b = 0$, $a \vee c = 1$, and $a \wedge c = 0$. Hence, $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. Then, by Theorem 3.2.42, it follows that $b = c$. ■

We now introduce the definition of *Boolean algebra*, named after the famous mathematician George Boole (1813–1864). Boole tried to formalize the process of logical reasoning using symbols instead of words. There are several equivalent definitions of Boolean algebra. Here we define Boolean algebra with the help of a lattice.

DEFINITION 3.2.48 ▶ A distributive lattice $(L, \leq)$ with the greatest element 1 and the least element 0 is called a **Boolean algebra** if every element has a complement in L.

From the above theorem, it follows that in a Boolean algebra $(L, \leq)$ every element $a \in L$ has a unique complement. The complement of a in L is denoted by a'.

EXAMPLE 3.2.49

Let $P(S)$ be the set of all subsets of a nonempty set S. Then $(P(S), \leq)$ is a poset, where $A \leq B$ if and only if $A \subseteq B$, for all $A, B \in P(S)$. This poset is a lattice, where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$, for all $A, B \in P(S)$. The subset S is the greatest element 1, and the empty subset $\emptyset$ is the least element 0. Also, for each $A \in P(S)$, the set complement of A in S is the complement of A in this lattice. Hence, the lattice $(P(S), \leq)$ is a Boolean algebra. We will discuss more about Boolean algebra in Chapter 12.

WORKED-OUT EXERCISES

Exercise 1: For each of the following relations, draw the digraph. Determine which are antisymmetric. Also determine which are partial orders.

(a) (S, R), where $S = \{2, 6, 8, 10, 20\}$ and R denotes the divisibility relation

(b) (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation

$$R = \{(1, 1), (5, 5), (6, 6), (8, 8), (10, 10), (1, 5), (5, 6), (1, 6)\}$$

(c) (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation

$$R = \{(1, 1), (5, 5), (6, 6), (8, 8), (10, 10), (1, 6)(8, 6), (6, 1)\}$$

Solution:

(a) Here $R = \{(2, 2), (6, 6), (8, 8), (10, 10), (20, 20), (2, 6), (2, 8), (2, 10), (2, 20), (10, 20)\}$. Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. There are no distinct elements $a, b \in S$ such that $(a, b) \in R$ and $(b, a) \in R$. Hence, the relation is antisymmetric. The relation is also transitive, because if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in S$. Hence, the relation R is a partial order. The digraph of this relation is shown in Figure 3.27(a).

(b) Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. There are no distinct elements $a, b \in S$ such that $(a, b) \in R$ and $(b, a) \in R$. Hence, the relation is antisymmetric. The relation is also transitive, because if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in S$. Hence, the relation R is a partial order. The digraph of this relation is shown in Figure 3.27(b).

(c) Because $(a, a) \in R$, for all $a \in S$, the relation is reflexive. This relation is not antisymmetric, because $(1, 6), (6, 1) \in R$, but $1 \neq 6$. Hence, this relation is also not a partial order. The digraph of this relation is shown in Figure 3.27(c).

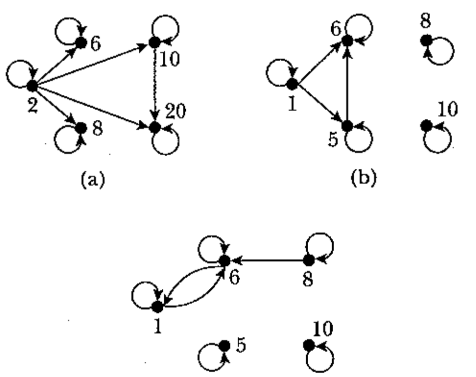


FIGURE 3.27 Digraphs

Exercise 2: Let $S = \{1, 2, 3\}$ and T be the set of all proper nonempty subsets of S. In the poset $(T, \leq)$, where $\leq$ is the set inclusion relation. Draw the digraph of the relation $\leq$ and the Hasse diagram of the poset. Find the maximal and minimal elements.

Solution: The digraph is shown in the Figure 3.28(a), and the Hasse diagram is shown in Figure 3.28(b).

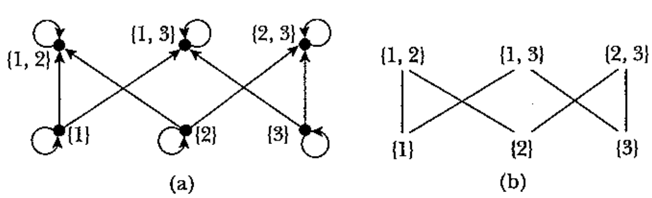


FIGURE 3.28 Digraph and Hasse diagram of $S = \{1, 2, 3\}$

In this poset, $\{1\}, \{2\}$, and $\{3\}$ are minimal elements, and $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are maximal elements.

Exercise 3: Draw the digraph of the divisibility relation and the Hasse Diagram of the poset $(D_{20}, \leq)$.

Solution: We have

$$D_{20} = \{1, 2, 4, 5, 10, 20\}.$$

The digraph is shown in Figure 3.29(a), and the Hasse diagram is shown in Figure 3.29(b).

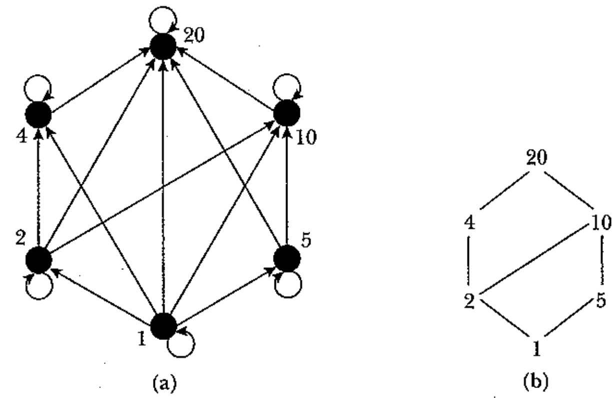


FIGURE 3.29 Digraph and Hasse diagram of D_{20}

Exercise 4: Define a relation R on the set $\mathbb{Z}$ of all integers by $m\,R\,n$ if and only if $m^2 = n^2$. Is R a partial order?

Solution: Because $m^2 = m^2$ for all $m \in \mathbb{Z}$, it follows that the relation is reflexive. Now $(2)^2 = (-2)^2$ implies that $2\,R\,(-2)$ and $(-2)\,R\,2$, but $-2 \neq 2$. Hence, R is not antisymmetric, and therefore R is not a partial order.

Exercise 5: Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two posets, where $S_1 = \{1, 2, 4\}$, $S_2 = \{1, 2, 3, 6\}$, and both the relations $\leq_1, \leq_2$ are divisibility relations. With respect to lexicographic order on $S_1 \times S_2$, find all pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (2, 3)$.

Solution: Note that $(a, b) \preceq (c, d)$ if and only if $a <_1 c$ or $a = c$ and $b \leq_2 d$. We find those pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (2, 3)$. The pairs are $(1, b) \in S_1 \times S_2$, $b \in S_2$ and $(2, b) \in S_1 \times S_2$ such that b divides 3. Hence, the pairs are $(1, 1), (1, 2), (1, 3), (1, 6), (2, 1)$, and $(2, 3)$.

Exercise 6: Consider the poset $(S, \leq)$, where $S = \{2, 4, 3, 6, 12\}$ and the partial order is the divisibility relation. Find a linear order on S compatible with the given partial order.

Solution: 2 is a minimal element of S. Let $a_1 = 2$ and $S_1 = S - \{2\} = \{4, 3, 6, 12\}$. Then S_1 is also a poset under the divisibility relation. Also, S_1 has a minimal element $a_2 = 3$. Let

$$S_2 = S - \{2, 3\} = \{4, 6, 12\}.$$

Now S_2 has a minimal element $a_3 = 4$. Let

$$S_3 = S - \{2, 3, 4\} = \{6, 12\}.$$

S_3 has a minimal element $a_4 = 6$. Let

$$S_4 = S - \{2, 3, 4, 6\} = \{12\}.$$

Finally, $a_5 = 12$ is a minimal element of $\{12\}$. We now define the partial order $\leq_1$ on S by $2 \leq_1 3 \leq_1 4 \leq_1 6 \leq_1 12$. It follows that this is a linear order.

Notice that $4 \leq 12$ because 4 divides 12. In the relation $\leq_1$, we have

$$4 \leq_1 6 \leq_1 12,$$

which implies that $4 \leq_1 12$ by the property of transitivity. Similarly, we can verify that the compatibility holds for other elements. So it follows that the linear order $\leq_1$ on S is compatible with the relation $\leq$.

Exercise 7: Show that every chain is a distributive lattice.

Solution: Let $(L, \leq)$ be a chain and $a, b, c \in L$. Because L is a chain, either $a \leq b$ or $b \leq a$. If $a \leq b$, then $a \vee b = b$ and $a \wedge b = a$. If $b \leq a$, then $a \vee b = a$ and $a \wedge b = b$. Hence, for any two elements $a, b \in L$, $a \wedge b$ and $a \vee b$ exist in L. Suppose $a \leq b$.

Case 1: $b \leq c$
Now $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. Hence, we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Case 2: $c \leq b$

Subcase 2a: $a \leq c$
In this case, we have $a \leq c \leq b$. Now $a \wedge (b \vee c) = a \wedge b = a$ and

$$(a \wedge b) \vee (a \wedge c) = a \vee a = a.$$

Hence,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Subcase 2b: $c \leq a$
In this case, we have $c \leq a \leq b$. Now $a \wedge (b \vee c) = a \wedge b = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee c = a$. Hence,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Similarly, if $b \leq a$, then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Exercise 8: In a lattice $(L, \leq)$, prove that $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee (a \wedge c))$ for all $a, b, c \in L$.

Solution: Now $a \wedge b \leq a$, $a \wedge c \leq a$. Therefore, $(a \wedge b) \vee (a \wedge c) \leq a$. Again, $a \wedge b \leq b$ implies

$$(a \wedge b) \vee (a \wedge c) \leq b \vee (a \wedge c).$$

Thus, we find that $(a \wedge b) \vee (a \wedge c)$ is a lower bound of $\{a, b \vee (a \wedge c)\}$. But $a \wedge (b \vee (a \wedge c))$ is the glb of $\{a, b \vee (a \wedge c)\}$. Hence,

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee (a \wedge c)).$$

Exercise 9: Consider the lattice $(D_{20}, \leq)$, where $\leq$ denotes the divisibility relation. Find $4 \wedge (5 \vee 10)$ and $(2 \vee (2 \wedge 5)) \vee 4$. Is this lattice a Boolean algebra?

Solution: $D_{20} = \{1, 2, 4, 5, 10, 20\}$. Now

$$\begin{aligned} 4 \wedge (5 \vee 10) &= 4 \wedge 10 && \text{because } 5 \vee 10 = \text{lcm}\{5, 10\} = 10 \\ &= 2 && \text{because } 4 \wedge 10 = \gcd\{4, 10\} = 2. \end{aligned}$$

Also,

$$\begin{aligned} (2 \vee (2 \wedge 5)) \vee 4 &= (2 \vee 1) \vee 4 && \text{because } 2 \wedge 5 = \gcd\{2, 5\} = 1 \\ &= 2 \vee 4 && \text{because } 2 \vee 1 = \text{lcm}\{2, 1\} = 2 \\ &= 4 && \text{because } 2 \vee 4 = \text{lcm}\{2, 4\} = 4. \end{aligned}$$

The Hasse diagram of D_{20} is shown in Figure 3.29. In this lattice, the least element is 1 and the greatest element is 20. Now,

$$\begin{array}{lll} 2 \wedge 1 = 1, & 2 \vee 1 \neq 20, & 2 \wedge 4 \neq 1, \\ 2 \wedge 5 = 1, & 2 \vee 5 \neq 20, & 2 \wedge 10 \neq 1. \end{array}$$

Hence, 2 has no complement in D_{20}. Therefore, this is not a Boolean algebra.

Exercise 10: Consider the lattice $(S, \leq)$, where $S = \{1, 2, 4, 5, 8, 9\}$ and $\leq$ denotes the usual "less than or equality" relation. Find $4 \wedge (5 \vee 9)$ and $(2 \vee (2 \wedge 8)) \vee 4$. Is this lattice a Boolean algebra?

Solution: In this lattice, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. The Hasse diagram of S is the given in Figure 3.30.

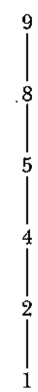


FIGURE 3.30
Hasse diagram of S

Now,

$$\begin{aligned} &4 \wedge (5 \vee 9) \\ &= 4 \wedge \max\{5, 9\} \\ &= 4 \wedge 9 \\ &= \min\{4, 9\} \\ &= 4 \end{aligned} \quad \text{and} \quad \begin{aligned} &(2 \vee (2 \wedge 8)) \vee 4 \\ &= (2 \vee \min\{2, 8\}) \vee 4 \\ &= (2 \vee 2) \vee 4 \\ &= \max\{2, 2\} \vee 4 \\ &= \max\{2, 4\} \\ &= 4. \end{aligned}$$

This is a chain. Hence, S is a distributive lattice with the greatest element 9 and the least element 1. In this lattice, suppose there exists an element b such that $2 \vee b = 9$ and $2 \wedge b = 1$. Then $\max\{2, b\} = 9$ implies that $b = 9$. On the other hand, $\min\{2, b\} = 1$ implies that $b = 1$. Thus, we find that 2 has no complement in this lattice. Hence, S is not a Boolean algebra.

SECTION REVIEW

Key Terms

antisymmetric
partial order
partially ordered set
poset
dual
comparable
linearly ordered set
totally ordered set
chain
product partial order
lexicographic order
dictionary order
closed
covers
Hasse diagram
minimal element
maximal element
greatest element
least element
compatible
topological ordering
upper bound
least upper bound (lub)
lower bound
greatest lower bound (glb)
lattice
distributive
complement
Boolean algebra

Some Key Definitions

1. A relation R on a set S is called antisymmetric if for all $a, b \in S$, $a\,R\,b$ and $b\,R\,a$, then $a = b$.
2. A relation R on a set A is called a partial order on A if R is reflexive, antisymmetric, and transitive.
3. A set A together with a partial order relation R is called a partially ordered set, or simply poset, and we denote this poset by (A, R).
4. Let $(S, \leq)$ be a poset and $a, b \in S$. If either $a \leq b$ or $b \leq a$, then we say that a and b are comparable. The poset $(S, \leq)$ is called a linearly ordered set, or a totally ordered set, or a chain, if for all $a, b \in S$ either $a \leq b$ or $b \leq a$.
5. Let $(S, \leq)$ be poset. An element $a \in S$ is called
   (i) a minimal element if there is no element $b \in S$ such that $b < a$,
   (ii) a maximal element if there is no element $b \in S$ such that $a < b$,
   (iii) a greatest element if $b \leq a$ for all $b \in S$,
   (iv) a least element if $a \leq b$ for all $b \in S$.
6. Let $(S, \leq)$ be a poset and let $\{a, b\}$ be a subset of S. An element $c \in S$ is called an upper bound of $\{a, b\}$ if $a \leq c$ and $b \leq c$.
7. An element $d \in S$ is called a least upper bound (lub) of $\{a, b\}$ if
   (i) d is an upper bound of $\{a, b\}$; and
   (ii) if $c \in S$ is an upper bound of $\{a, b\}$, then $d \leq c$.
8. Let $(S, \leq)$ be a poset and let $\{a, b\}$ be a subset of S. An element $c \in S$ is called a lower bound of $\{a, b\}$ if $c \leq a$ and $c \leq b$. An element $d \in S$ is called a greatest lower bound (glb) of $\{a, b\}$ if
   (i) d is a lower bound of $\{a, b\}$; and
   (ii) if $c \in S$ is a lower bound of $\{a, b\}$, then $c \leq d$.
9. A poset $(L, \leq)$ is called a lattice if $a \wedge b$ and $a \vee b$ exist in L for all $a, b \in L$.

10. A lattice $(L, \leq)$ is called distributive if it satisfies

(D1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

Some Key Results

1. Let $(S, \leq)$ be a poset such that S is a finite nonempty set. Then this poset has a minimal element.
2. In a poset $(S, \leq)$, if a subset $\{a, b\}$ of S has a lub, then this lub is unique.
3. In a poset $(S, \leq)$, if a subset $\{a, b\}$ of S has a glb, then this glb is unique.
4. Let $(L, \leq)$ be a lattice and $a, b, c \in L$. Then

   (L1) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

   (L2) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

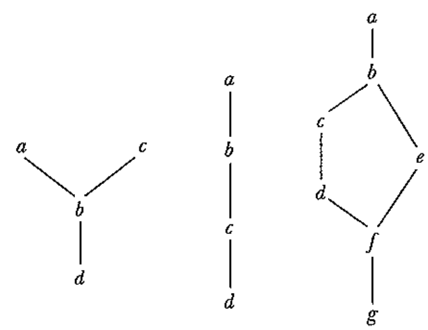
   (L3) $a \vee a = a$, $a \wedge a = a$,

   (L4) $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.

EXERCISES

1. For each of the following relations draw the digraph. Determine which relations are antisymmetric.
   a. (S, R), where $S = \{5, 6, 8, 10, 20\}$ and R denotes the divisibility relation
   b. (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation
   $$R = \{(1,1), (5,5), (6,6), (8,8), (10,10), (1,5), (5,6), (1,6)\}$$
   c. (S, R), where $S = \{1, 5, 6, 8, 10\}$ and R denotes the relation
   $$R = \{(1,1), (5,5), (6,6), (8,8), (10,10), (1,5), (8,6), (1,6)\}$$
2. Determine which of the following relations are antisymmetric.
   a. (S, R), where $S = \mathbb{Z}$, $a\,R\,b$ if and only if $a = b^n$ for some positive integer n
   b. (S, R), where $S = \mathbb{Z}$, $a\,R\,b$ if and only if $a = nb$ for some positive integer n
3. Define a relation R on the set $\mathbb{Z}$ of all integers by $m\,R\,n$ if and only if $|m| = |n|$. Is R a partial order?
4. Define a relation R on the set $\mathbb{Z}$ of all integers by $m\,R\,n$ if and only if $mn \geq 0$. Is R a partial order?
5. Define a relation R on the set $\mathbb{Z} \times \mathbb{Z}$ by $(m, t)\,R\,(n, r)$ if and only if $m = n$ and $t - r \geq 0$. Is R antisymmetric?
6. Draw the Hasse diagram for each of the following posets.
   a. $(\{a \mid a$ is a positive divisor of $20\}, \leq)$, where $\leq$ denotes the divisibility relation
   b. $(\mathbb{N}, \leq)$, where $\leq$ denotes the natural order relation
   c. $(\mathcal{P}(S), \leq)$, $S = \{1, 2, 3, 4\}$, where $\leq$ denotes the set inclusion relation
   d. $(\mathcal{P}(S) - \{\emptyset\}, \leq)$, $S = \{1, 2, 3\}$, where $\leq$ denotes the set inclusion relation
7. Let $S = \{1, 2, 3\}$ and
   $$A = \{\{2\}, \{3\}, \{2, 3\}, \{1, 3\}, S\}.$$
   Draw the digraph of the partial order $\leq$ defined by the set inclusion $\subseteq$ on A. Also draw the Hasse diagram of the poset $(A, \leq)$. Find all maximal and minimal elements of this poset.
8. Let $S = \{1, 2, 3, 6, 9, 18\}$. Consider the partial order $\leq$ defined by the divisibility relation on S. Draw the digraph of this partial order and the Hasse diagram of the poset $(S, \leq)$. Find all maximal and minimal elements of this poset.
9. Give an example of a relation R that is antisymmetric, but not reflexive.
10. Give an example of a poset $(P, \leq)$ such that P has two elements a and b for which $a \wedge b$ does not exist.
11. Show that $(\mathbb{R}, \leq)$ is not a poset, where $a \leq b$ means that $b = ad$ for some $d \in \mathbb{R}$.
12. Let A be the set of first 12 positive integers. Define a relation R on A by $x\,R\,y$ if and only if x is a divisor of y for all $x, y \in A$. Prove that (A, R) is a poset.
13. Define the relation $\geq$ on the set $\mathbb{C}$ of complex numbers by $a + ib \geq c + id$ if and only if $a \geq c$ and $b \geq d$ for all $a, b, c, d \in \mathbb{R}$. Prove that $\geq$ is a partial order on the set of complex numbers $\mathbb{C}$. Is it a linear order on $\mathbb{C}$? Justify your answer.
14. Let $\leq_1$ and $\leq_2$ be two partial orders on a set S. Is $\leq_1 \cap \leq_2$ a partial order on S?
15. Let $(A, \leq)$ and $(B, \leq)$ be two posets. Prove that $(A \times B, \leq)$ is a poset, where $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$.

16. Let $(A, \leq)$ and $(B, \leq)$ be two posets. Prove that $(A \times B, \preceq)$ is a poset, where $\preceq$ denotes lexicographic order. If $(A, \leq)$ and $(B, \leq)$ are linearly ordered sets, then prove that $(A \times B, \preceq)$ is a linearly ordered set.
17. Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two posets, where $S_1 = \{1, 2, 4\}$, $S_2 = \{1, 2, 3, 6\}$, and both the relations $\leq_1, \leq_2$ are divisibility relations. With respect to lexicographic order on $S_1 \times S_2$, find all pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (4, 2)$.
18. Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two posets, where $S_1 = \{1, 2, 4\}$, $S_2 = \{1, 2, 3, 6\}$, and both the relations $\leq_1, \leq_2$ are the usual "less than or equal to" relations. With respect to lexicographic order on $S_1 \times S_2$, find all pairs $(a, b) \in S_1 \times S_2$ such that $(a, b) \preceq (2, 3)$.
19. Consider the poset $(\{2, 3, 6, 12\}, |)$, where the partial order $|$ is the divisibility relation. Find a linear order $\leq$ on $\{2, 3, 6, 12\}$ such that $\leq$ is compatible with $|$.
20. Find a compatible linear order for the poset $(\{3, 6, 7, 12, 14, 18, 21\}, |)$, where the partial order $|$ is the divisibility relation.
21. Arrange the following words according to dictionary order.
    a. real, relation, relative, reliable, reason, invitation, invite
    b. poset, pond, posses, party, partial, orange, organize
22. Consider the poset $(S, \leq)$, where $S = \{m \mid m$ is a positive divisor of 48 and $1 < m < 48\}$ and the relation $\leq$ is the divisibility relation.
    a. Find all minimal and maximal elements.
    b. Find all lower bounds of $\{12, 16\}$.
    c. Find all upper bounds of $\{12, 16\}$.
    d. Find the glb and lub of $\{12, 16\}$.
    e. Does this poset contain the least element and the greatest element?
    f. Is this poset a lattice?
23. Consider the lattice $(S, \leq)$, where $S = \{3, 4, 5, 8, 9, 10\}$ $\leq$ denotes the usual "less than or equal to" relation. Find $4 \wedge (5 \vee 9)$ and $(3 \vee (3 \wedge 8)) \vee 4$. Is this lattice a Boolean algebra?
24. Let $S = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Consider the usual order relation on $\mathbb{Z}$. In the poset $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \preceq)$, arrange the following elements in increasing order according to the lexicographic order $\preceq$.
    a. $(1, 0, 5)$, $(0, 9, 0)$, $(3, -3, 5)$, $(-4, 9, 2)$, $(0, 0.2)$, $(54, 123, -312)$
    b. $(1, 0, 0)$, $(1, 0, 1)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$
25. Justify by examples.
    a. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may not have an upper bound.
    b. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may have more than one upper bound.
    c. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may not have a lub.
26. Justify by examples.
    a. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may not have a lower bound.
    b. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may have more than one lower bound.
    c. In a poset $(S, \leq)$, a subset $\{a, b\}$ of S may not have a glb.
27. Which of the posets in Figure 3.31 are lattices?



(a) (b) (c)

FIGURE 3.31 Posets

28. Let D_{40} denote the set of all positive divisors of 40. Consider the lattice

$$(D_{40}, \leq),$$

where $\leq$ denotes the divisibility relation. Find $4 \wedge (8 \vee 10)$ and $(2 \vee (2 \wedge 8)) \vee 20$.
29. Let D_{42} denote the set of all positive divisors of 42. Consider the lattice

$$(D_{42}, \leq),$$

where $\leq$ denotes the divisibility relation. Find $4 \wedge (6 \vee 14)$ and $(2 \vee (2 \wedge 8)) \vee 21$.
30. In a lattice $(L, \leq)$, prove the following. For all $a, b, c \in L$,
    a. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
    b. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$
    c. $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
    d. if $a \leq c$, then $a \vee (b \wedge c) \leq (a \vee b) \wedge c$
31. A lattice $(L, \leq)$ is called a **modular lattice** if for all $a, b, c \in L$, $a \leq c$ implies

$$a \vee (b \wedge c) = (a \vee b) \wedge c.$$

In a modular lattice $(L, \leq)$, prove that for all $a, b, c \in L$, $a \leq c$, $a \wedge b = c \wedge b$, and $a \vee b = c \vee b$ imply that $a = c$.
32. Prove that every distributive lattice is a modular lattice.
33. Give an example of a lattice that is modular but not distributive.
34. Prove that a lattice $(L, \leq)$ is distributive if and only if for all $a, b, c \in L$,

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

35. Determine whether the following assertions are true or false. If true, prove the result; if false, give a counterexample.
 a. The relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid |a - b| \leq 1\}$ is a partial order on $\mathbb{Z}$.
 b. The relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid |a| \leq |b|\}$ is a partial order on $\mathbb{Z}$.
 c. The relation $R = \{(a, b) \in S \times S \mid a \text{ divides } b \text{ in } \mathbb{N}\}$ is a partial order on $S = \{1, 2, 3, 4, 6, 12\}$.

3.3 APPLICATION: RELATIONAL DATABASE

In the preceding sections, we discussed relations in detail. We now describe an application of relations to database theory.

In the business world, generating relevant information in a timely manner is crucial to the success of a business. For example, stockbrokers rely on stock-market reports to buy, sell, and hold stocks. Students rely on mid-semester grade reports to decide which subjects need more study. To produce relevant information efficiently, we need quick access to data (raw facts) from which to generate appropriate information. Therefore, data collection, storage, and retrieval are some of the most important activities of an organization.

Managing data efficiently requires the use of a computer database, especially if we are dealing with large quantities of data. A **database** is a shared and integrated computer structure that stores

- end-user data; i.e., raw facts that are of interest to the end user;
- **metadata**, i.e., data about data through which data are integrated.

We can think of a database as a well-organized electronic file cabinet whose contents are managed by software known as a **database management system**; that is, a collection of programs to manage the data and control the accessibility of the data.

Consider the following table which lists various facts associated with students.

Table: Student

ID	Name	Rank	Major	EmpID
3456	Peter	Sr	CSC	745
9324	Ashley	Soph	Math	848
8723	Randy	Jr	CSC	745
2367	Sheila	Sr	Arts	467
8236	Anita	Fr	Drama	848
7623	Jackson	Sr	Math	848

The column headings are ID, Name, Rank, Major, and EmpID. Each row in the table describes a student's ID, name, college standing, major, and the ID of the advisor. The rows, among other things, specify which ID is associated with, i.e., related to, which student and the advisor of each student. In other words, we can think of each row as describing the relationships between ID, name, rank, major, and advisor's ID. Let us think of column headings as sets consisting of elements