(c) False.

Let \( U = \{a, b, c, d, e, f, g\} \), \( A = \{a, b, e, f\} \), and \( B = \{b, f, g\} \). Then,

\[ A - B = \{a, e\} \quad \text{and} \quad B - A = \{g\}. \]

Now,

\[ (A - B)' = U - (A - B) = \{b, c, d, f, g\} \]

and

\[ (B - A)' = U - (B - A) = \{a, b, c, d, e, f\}. \]

It follows that

\[ (A - B)' \neq (B - A)'. \]

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**SECTION REVIEW**

**Key Terms**

- set
- roster method
- set-builder method
- subset
- superset
- proper subset
- equal sets
- empty (null) set
- finite set
- infinite set
- singleton set
- power set
- universal set
- Venn diagrams
- union of sets
- intersection of sets
- disjoint sets
- index set
- set difference
- mutually disjoint
- pairwise disjoint
- relative complement
- complement of a set
- symmetric difference
- ordered pair
- Cartesian product
- diagonal of a set
- ordered n-tuples
- n-fold Cartesian product
- bit string
- length

**Some Key Definitions**

1. A set is a well-defined collection of objects.
2. Let \( X \) and \( Y \) be sets. Then \( X \) is said to be a subset of \( Y \), written \( X \subseteq Y \), if every element of \( X \) is an element of \( Y \).
3. Two sets \( X \) and \( Y \) are said to be equal, written \( X = Y \), if every element of \( X \) is an element of \( Y \) and every element of \( Y \) is an element of \( X \); i.e., if \( X \subseteq Y \) and \( Y \subseteq X \).
4. If there exists a nonnegative integer \( n \) such that \( X \) has \( n \) elements, then \( X \) is called a finite set with \( n \) elements; otherwise \( X \) is called an infinite set.
5. For any set \( X \), the power set of \( X \), written \( \mathcal{P}(X) \), is the set of all subsets of \( X \).
6. The union of two sets \( X \) and \( Y \), denoted by \( X \cup Y \), is defined to be the set \( X \cup Y = \{x \mid x \in X \text{ or } x \in Y\} \).
7. The intersection of two sets \( X \) and \( Y \), denoted by \( X \cap Y \), is defined to be the set \( X \cap Y = \{x \mid x \in X \text{ and } x \in Y\} \).
8. Two sets \( X \) and \( Y \) are said to be disjoint if \( X \cap Y = \emptyset \).
9. Let \( X \) and \( Y \) be sets. The difference of \( X \) and \( Y \) (or the relative complement of \( Y \) in \( X \)), written \( X - Y \), is the set \( X - Y = \{x \mid x \in X \text{ but } x \notin Y\} \).
10. The Cartesian product of two nonempty sets \( X \) and \( Y \), denoted by \( X \times Y \), is the set \( X \times Y = \{(x, y) \mid x \in X, y \in Y\} \).
Some Key Results

1. Let $X$ and $Y$ be sets. Then $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.
2. Let $X$ and $Y$ be sets. Then $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.
3. Let $X$, $Y$, $Z$ be subsets of a set $U$. Then
   
   (i) If $X \subseteq Y$, then $X \cup Y = Y$ and $X \cap Y = X$.
   (ii) $X \cup \emptyset = X$ and $X \cap \emptyset = \emptyset$.
   (iii) $X \cup X = X$ and $X \cap X = X$.
   (iv) $X \cup Y = Y \cup X$ and $X \cap Y = Y \cap X$.
   (v) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ and $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.
   (vi) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ and $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.
   (vii) $X \cap (X \cup Y) = X$ and $X \cup (X \cap Y) = X$.

4. Let $X$ and $Y$ be sets and $U$ be a universal set under consideration. Then
   
   (i) $X \cup X' = U$ and $X \cap X' = \emptyset$.
   (ii) $(X')' = X$.
   (iii) $X \cap Y = X \cap Y'$.
   (iv) (DeMorgan’s laws) $(X \cup Y)' = X' \cap Y'$ and $(X \cap Y)' = X' \cup Y'$.

EXERCISES

1. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{x \in \mathbb{Z} \mid x$ is divisible by $6\}$, and $C = \{x \in \mathbb{R} \mid x^2 = 2 \text{ or } x^2 = 1\}$. Mark the following true or false.
   
   a. $3 \in A$
   b. $6 \in A$
   c. $2 \notin A$
   d. $2 \in B$
   e. $6 \in B$
   f. $24 \in B$
   g. $28 \notin B$
   h. $2 \in C$
   i. $1 \in C$
   j. $-\sqrt{2} \in C$
   k. $5 \in A \cup B$
   l. $6 \in A \cap B$
   m. $1 \in A \cap C$
   n. $\sqrt{2} \in B \cup C$

2. Mark the following true or false.
   
   a. $28 \in \mathbb{Z}$
   b. $-5 \in \mathbb{N}$
   c. $\sqrt{2} \notin \mathbb{Q} \cap \mathbb{R}$
   d. $\mathbb{Z} \cup \mathbb{Q} = \mathbb{R}$
   e. $\mathbb{R} \cap \mathbb{C} = \mathbb{R}$

3. Let $U = \{a, b, c, d, e, f, g\}$, $A = \{a, d, e, f\}$, and $B = \{b, e, g\}$ be sets, where $U$ acts as the universal set. Determine the following.
   
   a. $(A \cup B)'$
   b. $A \cap B$
   c. $A - B$
   d. $B - A$

4. Let $U$ be the set of all students in a college. Let $A$ be the set of students taking the discrete mathematics course and $B$ be the set of students taking the calculus course. Describe the following.
   
   a. $A \cup B$
   b. $A \cap B$
   c. $A - B$
   d. $B - A$
   e. $A'$

5. Let $P = \{x \in \mathbb{N} \mid 2 < x \leq 8\}$, $Q = \{x \in \mathbb{N} \mid 0 \leq x < 5\}$, $R = \{x \in \mathbb{N} \mid 1 \leq x < 10\}$. Let $U = \{x \in \mathbb{Z} \mid -2 \leq x < 12\}$ be the universal set. Determine the following.
   
   a. $P \cup R$
   b. $Q \cap R$
   c. $P \Delta R$
   d. $Q'$

6. Let $P$, $Q$, $R$, and $U$ be the same as in Exercise 5. Verify the following.
   
   a. $(P \cup Q)' = P' \cap Q'$
   b. $P \cap (P \cup R) = P$
   c. $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

7. Let $A = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$ and $B = \{x \in \mathbb{R} \mid 3 \leq x \leq 8\}$. Find $A \cup B, A \cap B, A - B, B - A$.

8. Determine whether the following pairs of sets are equal. Justify your answer.
   
   a. $A = \left\{ n \in \mathbb{Z} \mid n = \frac{1}{n^2} \right\}$
   b. $B = \{x \in \mathbb{R} \mid x^2 = 1\}$

9. Does every set have a subset? Give an example of a set that has only one proper subset.

10. Let $X$ be a set with 4 elements. Find $|P(X)|$.

11. Find $P(P(P(\emptyset)))$.

12. Let $n = \{1, 2, \ldots, n\}$, the set of first $n$ natural numbers.
   
   a. Describe the set $I_{10} - I_5$.
   b. Describe the set $I_n - I_m$ if
      
      (i) $n > m$
      (ii) $n = m$
      (iii) $n < m$

13. Let $A$ and $B$ be subsets of the set $U$. Draw the Venn diagram of the following sets.
   
   a. $(A \cup B)'$
   b. $(A \cap B)'$
   c. $A \Delta B$
   d. $(A \cup B) - (A \cap B)$

14. Let $A, B,$ and $C$ be subsets of the set $U$. Draw the Venn diagram of the following sets.
   
   a. $(A \cup C) \cap C$
   b. $(A \cap B) \cup C$
   c. $(A \cap B) - C$
   d. $(A - B) - C$
   e. $(A - (B \cup C)) \cup (B - (A \cup C))$
15. Let \( A, B, C, \) and \( D \) be subsets of the set \( U \). Draw the Venn diagram of the following sets.
   a. \( A \cap B \cap C \cap D \)
   b. \((A \cup B \cup C) \cap D\)
   c. \((A \cup B) \cap (C \cap D)\)

16. What sets do each of the Venn diagrams in Figure 1.10 represent?

   ![Venn Diagrams](image)

**Figure 1.10**

17. Let \( A \) and \( B \) be sets. Prove that \( A \subseteq B \) if and only if \( A \cap B = A \).

18. Prove those parts of Theorem 1.1.29 that are not proved in this section.

19. Suppose \( P \) and \( Q \) are two sets. Let \( R \) be a set that contains elements belonging to \( P \) or \( Q \) but not both. Let \( T \) be a set that contains elements belonging to \( Q \) or the complement of \( P \) but not both. Show that \( R \) is the complement of \( T \).

20. Let \( A \) and \( B \) be sets. Prove that \( A - (A - B) = A \cap B \).

21. Let \( A, B, \) and \( C \) be subsets of a set \( U \). Prove the following.
   a. \((A - B) \cup (A \cap B) \cap ((B - A) \cup (A \cup B)) = \emptyset\)
   b. \(A - (B \cap C) = (A - B) \cup (A - C)\)
   c. \(A - (B \cup C) = (A - B) \cap (A - C)\)
   d. \(A \cap (B - C) = (A \cap B) - (A \cap C)\)

22. Let \( U = \{a, b, c, d, e, f, g\}, A = \{a, e, f\}, B = \{b, g\}, \) and \( C = \{d, e, g\} \) be sets, where \( U \) acts as the universal set. Determine the following sets.
   a. \(A \times B\)
   b. \(B \times C\)
   c. \(A \times C\)
   d. \(A \times B \times C\)

23. Let \( U = \{a, b, c, d, e, f, g\}, A = \{a, d, e, f\}, B = \{b, e, g\}, \) and \( C = \{a, e, e, g\} \) be sets, where \( U \) acts as the universal set. Verify that
   \[A \times (B - C) = (A \times B) - (A \times C)\].

24. Let \( A, B, \) and \( C \) be sets. Prove the following.
   a. \(A \times (B \cap C) = (A \times B) \cap (A \times C)\)
   b. \(A \times (B \cup C) = (A \times B) \cup (A \times C)\)
   c. \(A \times B = \bigcup_{a \in B} A \times \{a\}\)

25. Let \( A \) and \( B \) be sets as in Exercise 23. Verify that \((A - B) \cup (B - A) = (A \cup B) - (A \cap B)\).

26. If \( A, B, \) and \( C \) are subsets of a set \( U \), then prove that \( A - C = B - C \) if and only if \( A \cup C = B \cup C \).

27. Let \( A, B, \) and \( C \) be sets. Prove that \(A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)\).

28. Let \( A \) and \( B \) be finite subsets of a set \( U \). Show that
   a. \(|A - B| = |A| - |A \cap B|\)
   b. \(|A \cup B| \leq |A| + |B|\). Moreover, show that \(|A \cup B| = |A| + |B|\) if and only if \(A \cap B = \emptyset\).
   c. \(|A \cup B| = |A| + |B| - |A \cap B|\).

29. In Figure 1.11, \( A \) is the set of people who go to a resort area for vacation, \( B \) is the set of people who take a cruise for vacation, and \( C \) is the set of people who go to a national park for vacation. The numbers in the figure represent the number of people in that region. Suppose that \(|A| = 150, |B| = 100, \) and \(|C| = 300\).

**Figure 1.11**

a. How many people only go to a resort area for vacation?
b. How many people only take a cruise for vacation?
c. How many people only go to a national park for vacation?
d. How many people either go to a resort area for vacation or take a cruise for vacation?
e. How many people use one of the three methods to take a vacation?

30. In Figure 1.12, \( A \) is the set of students taking algebra, \( B \) is the set of students who play basketball, \( C \) is the set of students taking the computer programming course, and the numbers in each region represent the number of students in that region.
32. Prove those parts of Theorem 1.1.38 that are not proved in this section.

33. Let $U$ be a universal set of 20 elements.
   a. What is the bit string of the empty set?
   b. What is the bit string of $U$?

34. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$. Let $A = \{2, 5, 8, 9, 10, 13\}$ and $B = \{1, 3, 7, 9, 12, 13\}$. Determine the following.
   a. $s_A$
   b. $s_A'$
   c. $s_B$
   d. $s_{A\cup B}$
   e. $s_{A\cap B}$

35. Let $U = \{x \in \mathbb{Z} | 1 \leq x \leq 30\}$. Let $A = \{x \in U | 2 \text{ divides } x\}$ and $B = \{x \in U | 3 \text{ divides } x\}$. Determine the following.
   a. $s_A$
   b. $s_A'$
   c. $s_B$
   d. $s_{A\cup B}$
   e. $s_{A\cap B}$

36. Let $U, P, Q, R$ and $R$ be as given in Exercise 5. Determine the following.
   a. $s_{P\cup Q}$
   b. $s_{P\cup R}$
   c. $s_{P\cup Q\cup R}$
   d. $s_{Q\cup R}$
   e. $s_{Q\cup Q\cup R}$
   f. $s_{P\cup Q\cup R}$
   g. $s_{P\cup Q\cup R\cup U}$

37. Prove the following set-theoretic statements if you find them correct or else give an example to disprove the result. The sets $A, B$ and $C$ are subsets of a set $U$.
   a. $A \cup (B - C) = (A \cup B) - (A \cup C)$
   b. $(A - B) - C = A - (B \cup C)$
   c. $(A \cup B) - A = A - B$
   d. $A - C = B - C$ if and only if $A \cup C = B \cup C$

1.2 Mathematical Logic

Intuitively, logic is the discipline that considers the methods of reasoning. It provides the rules and techniques for determining whether an argument is valid or not. In everyday life, we use reasoning to prove different points. For example, to prove to our parents that we passed an exam, we might show the test and the score. Or to prove to the utility company that our bill has been paid, we might show the cancelled check. Similarly, in mathematics and computer science, mathematical logic or logic is used to prove results. To be specific, in mathematics we use logic or logical reasoning to prove theorems, and in computer science we use logic or logical reasoning to prove the correctness of programs and also to prove theorems. (Mathematical logic is a discipline in its own right, and it is worthy of full treatment. In this section, however, we present and discuss only those aspects of mathematical logic that are necessary to the study of mathematics and computer science.)

Throughout the book not only are results given, but whenever a computer algorithm is available to solve the problem, an algorithm is presented. Therefore, our main objective is to use logic to prove theorems and the correctness of the programs. We therefore begin by defining the word theorem.

A theorem is a statement that can be shown to be true (under certain conditions). For example, in mathematics the following statement is a theorem,

If $x$ is an even integer, then $x + 1$ is an odd integer.