ECE 5/7383 Introduction to Quantum Informatics

Homework 2

Waves, Matter Waves, and the Schrödinger Equation

Assigned: ____________ Due: Monday, Sept. 28, 2026 at 2:00PM (CDT)

Work every problem by hand and show the intermediate steps; an answer with no supporting work receives no credit. Parts labeled *(for 7383 students only)* are required of ECE 7383 students and will receive NO CREDIT if answered by ECE 5383 students. This homework is worth 100 points total for both ECE 5383 and ECE 7383 students.

Throughout this assignment Dirac's "bra-ket" notation is used. A ket $|\psi\rangle$ denotes a column vector, the corresponding bra $\langle\psi|$ denotes its adjoint (conjugate transpose), which is a row vector, $\langle\varphi|\psi\rangle$ denotes the inner product, and $|\psi\rangle\langle\varphi|$ denotes the outer product, which is a matrix. The computational basis is $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$, and the Hadamard basis is $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. The Pauli operators are

$$\mathbf{X} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$$

A circumflex (sometimes I call it a "hat") such as $\widehat{\mathbf{H}}$ or $\hat{p}_x$ denotes an operator (note that in statistics, it denotes an estimator). The constant e is Euler's number, i satisfies $i^2 = -1$, and z^* denotes the complex conjugate of z. Unless a problem states otherwise, use the following values for the physical constants:

$$h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s}, \qquad \hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34}\ \mathrm{J \cdot s}, \qquad c = 2.998 \times 10^{8}\ \mathrm{m/s},$$

$$m_e = 9.109 \times 10^{-31}\ \mathrm{kg}, \qquad 1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}.$$

For a complex number z, $\arg(z)$ is its "argument," meaning the phase angle of z in the complex plane, measured counterclockwise from the positive real axis.

A state vector is **normalized** when its inner product with itself equals unity:

$$\langle\psi|\psi\rangle = 1, \text{ equivalently } \|\,|\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} = 1.$$

$\langle\psi \mid \psi\rangle = 1$, equivalently $\||\, \psi\rangle \| = \langle\psi \mid \psi\rangle = 1$.

You may submit your completed assignment to me electronically via an email attachment in the form of a PDF file (this is preferable), or you can turn in hardcopy at the beginning of class (2PM) of the due date. The email time and date stamp will serve as proof of your submission date and time for electronic submissions. Grading penalties for late assignments are in force. The grading policy for late assignments is posted on the class webpage/syllabus.

This homework is for your benefit and will help you to prepare for exams, so I hope you take it seriously.

1. **(Dirac bra-ket notation — 15 pts for 5383; 14 pts for 7383)**

Consider the unnormalized state $|\tilde{\psi}\rangle = 3|0\rangle + 4i|1\rangle$ together with the state $|\varphi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

a) Write $|\tilde{\psi}\rangle$ as a column vector and write the corresponding bra $\langle\tilde{\psi}|$ explicitly as a row vector. Compute $\langle\tilde{\psi}|\tilde{\psi}\rangle$ and give the normalized state $|\psi\rangle$.

b) Compute $\langle\varphi|\psi\rangle$ and $\langle\psi|\varphi\rangle$, and verify that $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$. Give $|\langle\varphi|\psi\rangle|^2$, which is the probability that a measurement of $|\psi\rangle$ in the Hadamard basis yields the outcome associated with $|+\rangle$.

c) Compute the outer product $\widehat{\mathbf{P}} = |\psi\rangle\langle\psi|$ as a 2×2 matrix. Verify that $\widehat{\mathbf{P}}$ is Hermitian, that $\widehat{\mathbf{P}}^2 = \widehat{\mathbf{P}}$, and that $\mathrm{Tr}(\widehat{\mathbf{P}}) = 1$. An operator with these properties is a "projector."

d) Compute the expectation values, $\langle\psi|\mathbf{Z}|\psi\rangle$ and $\langle\psi|\mathbf{X}|\psi\rangle$.

e) *(for 7383 students only)* Show that the computational basis satisfies the completeness relation $|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbf{I}$, and use that relation to show that any single-qubit state can be expanded as $|\psi\rangle = \langle 0|\psi\rangle|0\rangle + \langle 1|\psi\rangle|1\rangle$. Evaluate the two expansion coefficients for the $|\psi\rangle$ of part (a).

2. **(A linear differential equation with a harmonic solution — 15 pts for 5383; 14 pts for 7383)**

An undamped harmonic oscillator obeys the linear, homogeneous, second-order differential equation

$$\frac{d^2x(t)}{dt^2} + \omega_0^2\, x(t) = 0,$$

with $\omega_0 = 200$ rad/s. The initial conditions are $x(0) = 3$ cm and $\left.\frac{dx}{dt}\right|_{t=0} = 400$ cm/s.

a) Verify by direct substitution that $x(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$ satisfies the differential equation for any values of the constants c_1 and c_2. State why this two-parameter family is the general solution.

b) Apply the initial conditions to determine c_1 and c_2, and write $x(t)$.

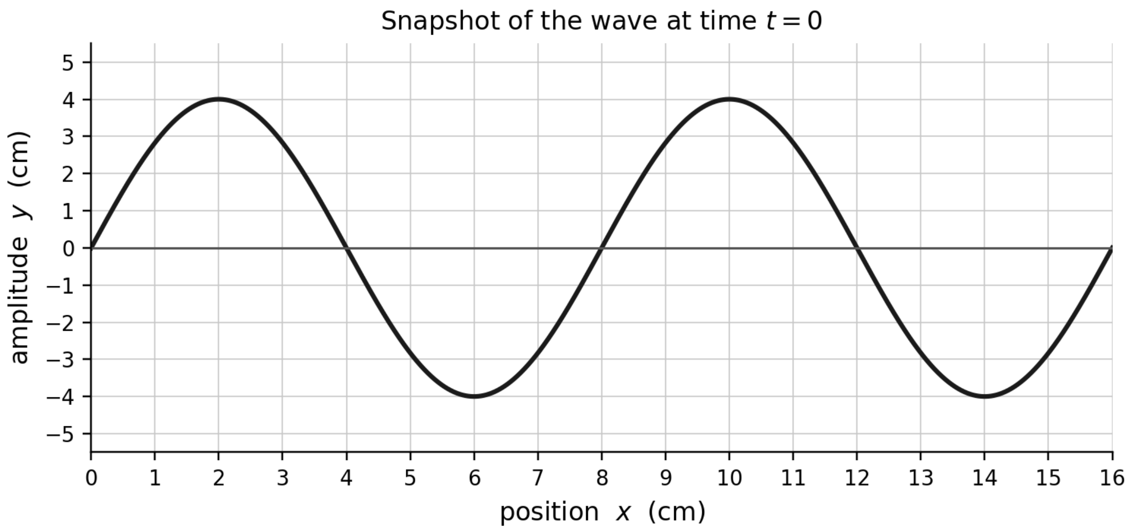
c) Show that $x(t)$ can be written in the amplitude-phase form $x(t) = R\cos(\omega_0 t - \phi)$. Derive the expressions for R and ϕ in terms of c_1 and c_2, and evaluate them numerically (give ϕ in both radians and degrees).

d) Give the temporal period T, the temporal frequency f in Hz, and the maximum speed of the oscillator.

e) *(for 7383 students only)* Show that the same solution can be written in the complex exponential (phasor) form $x(t) = \mathrm{Re}\{A\, e^{i\omega_0 t}\}$, and determine the complex constant A in both rectangular and polar form. State the physical meaning of $|A|$ and of $\arg(A)$.

3. **(Reading a travelling wave from its snapshot — 15 pts for 5383; 14 pts for 7383)**

The figure below is a snapshot, taken at time $t = 0$, of a sinusoidal wave that travels in the $+x$ direction. The horizontal axis is position in centimeters and the vertical axis is amplitude in centimeters. The wave has a temporal frequency of $f = 12$ Hz. Show the details of all calculations, including the equations used.



a) What is the amplitude, in cm?

b) What is the wavelength λ, in cm, and the wavenumber $k = 2\pi/\lambda$, in rad/cm?

c) What is the temporal period T, in seconds, and the angular frequency ω, in rad/s

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d) What is the propagation velocity of this wave, in cm/s and in m/s?

e) Write the travelling wave $y(x, t)$ explicitly, with x in cm and t in s, in a form consistent with both the snapshot and with propagation in the $+x$ direction.

f) *(for 7383 students only)* Suppose this same wave is one of the standing-wave harmonics of a string of length $L = 24$ cm that is fixed at both ends. Using the relation $\ell(\lambda/2) = L$ from the class notes, determine which harmonic ℓ this is, and state how many nodes the standing wave has, counting the two fixed ends.

4. **(Quantized light: Planck's constraint and the photoelectric effect — 14 pts for 5383; 13 pts for 7383)**

A continuous-wave laser emits at a free-space wavelength of $\lambda = 532$ nm with an optical output power of $P = 5.00$ mW.

a) Compute the frequency ν of the light and the energy of a single photon, $E = h\nu$, expressed both in joules and in electron-volts.

b) Planck's constraint on the energy contained in a mode of oscillation is $E = nh\nu$, where n is an integer. Using that idea, compute how many photons per second this laser emits.

c) The beam illuminates a photocathode whose work function is $\phi_{\text{work}} = 2.00$ eV. Compute the threshold frequency $\nu_0 = \phi_{\text{work}}/h$ and the corresponding threshold wavelength λ_0. Will photoelectrons be emitted at 532 nm? If so, compute the maximum kinetic energy of the emitted electrons in eV.

d) *(for 7383 students only)* Compute the stopping voltage V_0 that just prevents the most energetic photoelectrons from reaching the collector. Then explain, in two or three sentences, why doubling the optical power of the laser does not change V_0, and why that experimental observation cannot be accounted for by a purely classical wave description of light.

5. **(de Broglie matter waves — 13 pts for 5383; 13 pts for 7383)**

The de Broglie relation from the class notes states that a particle of momentum p has an associated wavelength:

$$p = mv = \frac{h}{\lambda}$$

a) An electron starts from rest and is accelerated through a potential difference of 100 V. Treating the motion as non-relativistic, compute its kinetic energy in joules, its momentum, and its de Broglie wavelength. Express the wavelength in nm. Comment briefly on the significance of the result.

b) A baseball of mass 0.145 kg moves at 40.0 m/s. Compute its de Broglie wavelength and compare it with a nuclear diameter of roughly 1×10^{-15} m. In one or two sentences, explain what this comparison says about the classical limit and the correspondence principle ($\lambda \to 0$) discussed in the class notes.

c) *(for 7383 students only)* de Broglie argued that an electron in a stable circular orbit of radius r must fit an integer number of its own wavelengths around the circumference, $n\lambda = 2\pi r$. Show that this standing-wave condition is equivalent to quantization of the orbital angular momentum in the form $L = rp = n\hbar$.

6. **(The time-independent Schrödinger equation: a particle in a box — 16 pts for 5383; 16 pts for 7383)**

An electron is confined to a one-dimensional box by the potential energy function $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) \to \infty$ outside that interval, so that $\psi(x) = 0$ everywhere outside the box. Inside the box the time-independent Schrödinger wave equation from the class notes is:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi(x) = 0$$

a) Show that inside the box the equation reduces to $\frac{d^2\psi}{dx^2} + k^2\psi = 0$ with $k = \sqrt{2mE}/\hbar$, and verify by substitution that $\psi(x) = A\cos(kx + \varphi)$ is a solution for any constants A and φ.

b) Apply the boundary conditions $\psi(0) = \psi(a) = 0$. Show that the first condition forces $\varphi = \pi/2$, so that $\psi(x) \propto \sin(kx)$, and that the second condition forces $ka = n\pi$ for $n = 1,2,3,\ldots$ Explain why $n = 0$ is excluded.

c) Show that the allowed energies are $E_n = \frac{n^2h^2}{8ma^2}$, and determine the normalization constant A from the requirement $\int_0^a |\psi(x)|^2\, dx = 1$.

d) Take $a = 0.70$ nm, which is a crude model of an electron delocalized along the backbone of a dye molecule. Compute E_1 and E_2 in eV, the transition energy $\Delta E = E_2 - E_1$, and the wavelength of the photon emitted in that transition. In what part of the electromagnetic spectrum does that wavelength fall?

e) *(for 7383 students only)* The position operator is $\hat{x}$, which multiplies a wave function by x, and the momentum operator from the class notes is $\hat{p}_x = -i\hbar\frac{d}{dx}$. Evaluate the commutator $[\hat{x}, \hat{p}_x]\psi(x) = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi(x)$ and show that it equals $i\hbar\psi(x)$. State what this implies about the simultaneous measurement of position and momentum.

7. **(The time-dependent Schrödinger equation and the stationary qubit — 12 pts for 5383; 16 pts for 7383)**

A qubit has the Hamiltonian operator $\hat{\mathbf{H}} = \frac{\hbar\omega}{2}\mathbf{Z}$, whose eigenvectors are the computational basis states with eigenvalues $E_0 = +\frac{\hbar\omega}{2}$ for $|0\rangle$ and $E_1 = -\frac{\hbar\omega}{2}$ for $|1\rangle$. The state of the qubit obeys the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle.$$

a) Suppose $|\psi(0)\rangle = |0\rangle$. Verify by substitution that $|\psi(t)\rangle = e^{-iE_0 t/\hbar}|0\rangle$ satisfies the time-dependent Schrödinger equation. Then show that for any operator $\hat{\mathbf{A}}$ the expectation value $\langle\psi(t)|\hat{\mathbf{A}}|\psi(t)\rangle$ does not depend on t, and explain why such a state is called a stationary state.

b) Now suppose $|\psi(0)\rangle = |+\rangle$. Write $|\psi(t)\rangle$. Show that the probabilities of measuring 0 and 1 in the computational basis are both $1/2$ for all t, but that the relative phase between the two components does evolve, and give the angular frequency at which it evolves.

c) Compute the expectation values $\langle\psi(t)|\mathbf{X}|\psi(t)\rangle$ and $\langle\psi(t)|\mathbf{Z}|\psi(t)\rangle$. Then determine the earliest time $t > 0$ at which the qubit reaches the state $|-\rangle$, up to a global phase.

d) *(for 7383 students only)* The formal solution of the time-dependent Schrödinger equation is $|\psi(t)\rangle = \hat{\mathbf{U}}(t)|\psi(0)\rangle$ with $\hat{\mathbf{U}}(t) = e^{-i\hat{\mathbf{H}}t/\hbar}$. Using $\mathbf{Z}^2 = \mathbf{I}$ and the matrix exponential identity established in Homework 1, show that $\hat{\mathbf{U}}(t) = \cos\left(\frac{\omega t}{2}\right)\mathbf{I} - i\sin\left(\frac{\omega t}{2}\right)\mathbf{Z}$. Write $\hat{\mathbf{U}}(t)$ explicitly as a 2×2 matrix, verify that $\hat{\mathbf{U}}^\dagger\hat{\mathbf{U}} = \mathbf{I}$ and that $\det\hat{\mathbf{U}} = 1$, and apply $\hat{\mathbf{U}}(t)$ to $|+\rangle$ to confirm your answer to part (b).