

ECE 5/7383 Introduction to Quantum Informatics

Homework 3

Complex Amplitudes, the Schrödinger Equation, Projective Measurement, and Information

Assigned: _____

Due: Monday, Oct. 19, 2026 at 2:00PM (CDT)

Work every problem by hand and show the intermediate steps; an answer with no supporting work receives no credit. Parts labeled (*for 7383 students only*) are required of ECE 7383 students and will receive NO CREDIT if answered by ECE 5383 students. This homework is worth 100 points total for both ECE 5383 and ECE 7383 students.

Dirac's "bra-ket" notation is used throughout. A ket $|\psi\rangle$ denotes a column vector, the bra $\langle\psi|$ denotes its adjoint, which is a row vector, $\langle\varphi|\psi\rangle$ denotes the inner product, and $|\psi\rangle\langle\varphi|$ denotes the outer product, which is a matrix. The computational basis is $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the Hadamard basis is $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The Pauli operators are

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A circumflex such as $\hat{\mathbf{H}}$ or $\hat{p}_x$ denotes an operator. The constant e is Euler's number, i satisfies $i^2 = -1$, and z^* denotes the complex conjugate of z . For a complex number z , $|z|$ is its modulus and $\arg(z)$ is its argument, the phase angle measured counterclockwise from the positive real axis, so that $z = |z|e^{i\arg(z)}$. A state $|\psi\rangle$ is normalized when $\langle\psi|\psi\rangle = 1$. Unless a problem states otherwise, use the following values for the physical constants:

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}, \quad \hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34} \text{ J} \cdot \text{s}, \quad m_e = 9.109 \times 10^{-31} \text{ kg}, \\ 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}.$$

You may submit your completed assignment to me electronically via an email attachment in the form of a PDF file (this is preferable), or you can turn in hardcopy at the beginning of class (2PM) of the due date. The email time and date stamp will serve as proof of your submission date and time for electronic submissions. Grading penalties for late assignments are in force. The grading policy for late assignments is posted on the class webpage/syllabus.

This homework is for your benefit and will help you to prepare for exams, so I hope you take it seriously.

1. (Algebra of complex amplitudes — 16 pts for 5383; 15 pts for 7383) Every amplitude in quantum informatics is a complex number, so fluency with complex arithmetic is a prerequisite for everything that follows. Let $z_1 = 3 + 4i$ and $z_2 = 1 - i$.

a) Express each of $z_1 + z_2$, $z_1 z_2$, and z_1/z_2 in the rectangular form $a + ib$ with a and b real. For the quotient, show the step in which you multiply by the conjugate of the denominator.

b) Give the modulus and the argument of z_1 and of z_2 , and write each in the polar form $|z|e^{i\arg(z)}$. Then verify, using your answer from part (a), that $|z_1 z_2| = |z_1||z_2|$ and that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

c) Let $z = a + ib$ with a and b real. Show by direct expansion that $z + z^* = 2 \operatorname{Re}(z)$ and that $zz^* = |z|^2$. Then use the second identity to show that $\langle \psi | \psi \rangle$ is real and non-negative for every single-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex.

d) (*for 7383 students only*) Show, by writing $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ and expanding both sides, that $(z_1 z_2)^* = z_1^* z_2^*$. Then use that result to prove that $\langle \varphi | \psi \rangle^* = \langle \psi | \varphi \rangle$ for any two single-qubit states.

2. (The one-dimensional Schrödinger equation for a free particle — 17 pts for 5383; 16 pts for 7383) The time-independent Schrödinger equation in one dimension is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x).$$

A free particle is one for which the potential energy is constant everywhere, and the constant may be taken to be zero, so $V(x) = 0$ for all x .

a) Show by direct substitution that $\psi(x) = Ae^{ikx}$, with A and k constants, satisfies the free-particle equation, and determine the energy E in terms of $\hbar$, m , and k .

b) The momentum operator is $\hat{p}_x = -i\hbar \frac{d}{dx}$. Show that $\psi(x) = Ae^{ikx}$ is also an eigenfunction of $\hat{p}_x$ and give its momentum eigenvalue. Then show that your answer to part (a) is consistent with the classical relation $E = p^2/2m$, and state the relationship between k and the de Broglie wavelength λ .

c) An electron is in the free-particle state above with $k = 1.00 \times 10^{10} \text{ m}^{-1}$. Compute its momentum, its energy in joules and in eV, and its de Broglie wavelength in nm.

d) Compute the probability density $|\psi(x)|^2$ for this state and describe in one sentence what it says about where the particle is located.

e) (*for 7383 students only*) Explain why this wave function cannot be normalized over the whole real line, and state what that implies about whether a particle can have an exactly defined momentum in practice.

3. (Projective measurement in two different bases — 18 pts for 5383; 17 pts for 7383) A qubit is prepared in the state

$$|\psi\rangle = \frac{1+2i}{\sqrt{14}}|0\rangle + \frac{3}{\sqrt{14}}|1\rangle.$$

a) Verify that $|\psi\rangle$ is normalized. Then write the computational-basis projectors $\hat{\mathbf{P}}_0 = |0\rangle\langle 0|$ and $\hat{\mathbf{P}}_1 = |1\rangle\langle 1|$ as 2×2 matrices, and verify the three defining properties $\hat{\mathbf{P}}_0 + \hat{\mathbf{P}}_1 = \mathbf{I}$, $\hat{\mathbf{P}}_j^2 = \hat{\mathbf{P}}_j$, and $\hat{\mathbf{P}}_0\hat{\mathbf{P}}_1 = \mathbf{0}$.

b) Compute the probabilities $P(0) = \langle\psi|\hat{\mathbf{P}}_0|\psi\rangle$ and $P(1) = \langle\psi|\hat{\mathbf{P}}_1|\psi\rangle$ of the two outcomes of a computational-basis measurement, and verify that they sum to one. Give the normalized post-measurement state when the outcome is 0.

c) Now consider measuring the same prepared state $|\psi\rangle$ in the Hadamard basis instead. Write the projectors $\hat{\mathbf{P}}_+ = |+\rangle\langle +|$ and $\hat{\mathbf{P}}_- = |-\rangle\langle -|$ as matrices, compute $P(+)$ and $P(-)$, and verify that they sum to one.

d) (*for 7383 students only*) The expectation value of an observable can be obtained from the measurement probabilities. Using your results from parts (b) and (c), compute $\langle\mathbf{Z}\rangle = P(0) - P(1)$ and $\langle\mathbf{X}\rangle = P(+)-P(-)$. Then verify both by evaluating $\langle\psi|\mathbf{Z}|\psi\rangle$ and $\langle\psi|\mathbf{X}|\psi\rangle$ directly as matrix products.

4. **(Projective measurement of photon polarization — 18 pts for 5383; 17 pts for 7383)** The polarization of a single photon is a qubit. Take horizontal polarization as $|H\rangle = |0\rangle$ and vertical polarization as $|V\rangle = |1\rangle$. A linear polarizer whose transmission axis is at angle θ from the horizontal performs a projective measurement onto the state

$$|\theta\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle.$$

A photon that passes through the polarizer emerges in the state $|\theta\rangle$, and a photon that is absorbed is simply lost.

a) Write the projector $\hat{\mathbf{P}}_\theta = |\theta\rangle\langle\theta|$ as a 2×2 matrix in terms of $\cos\theta$ and $\sin\theta$.

b) A photon is prepared in the state $|H\rangle$ and is sent through a single polarizer set at $\theta = 60^\circ$. Compute the probability that it is transmitted, and give the state of the photon after transmission. The general result you obtain here is known as Malus's law.

c) A photon prepared in $|H\rangle$ is sent toward a polarizer set at 90° . Compute the probability that it is transmitted. Now insert a second polarizer set at 45° *between* the source and the 90° polarizer, and compute the probability that the photon passes through both. Comment on the comparison, and explain why the result does not occur for a classical filter that only removes photons.

d) (*for 7383 students only*) Show that $\hat{\mathbf{P}}_\theta$ satisfies the three requirements of a projector associated with one outcome of a two-outcome measurement, namely that it is Hermitian, that $\hat{\mathbf{P}}_\theta^2 = \hat{\mathbf{P}}_\theta$, and that $\hat{\mathbf{P}}_\theta + \hat{\mathbf{P}}_{\theta+90^\circ} = \mathbf{I}$. State the physical meaning of the last identity.

5. (A simple Hamiltonian and its time evolution — 16 pts for 5383; 17 pts for 7383) A qubit has the Hamiltonian operator

$$\hat{\mathbf{H}} = \hbar\omega|1\rangle\langle 1| = \hbar\omega \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where ω is a real positive constant. The state obeys the time-dependent Schrödinger equation $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$.

a) Verify that $\hat{\mathbf{H}}$ is Hermitian. Show that $|0\rangle$ and $|1\rangle$ are eigenvectors and give the corresponding energy eigenvalues. Then give the expectation value $\langle \hat{\mathbf{H}} \rangle$ for a general normalized state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

b) The time evolution operator is $\hat{\mathbf{U}}(t) = e^{-i\hat{\mathbf{H}}t/\hbar}$. Because $\hat{\mathbf{H}}$ is diagonal, the exponential may be taken entry by entry on the diagonal. Write $\hat{\mathbf{U}}(t)$ explicitly as a 2×2 matrix and verify that $\hat{\mathbf{U}}^\dagger \hat{\mathbf{U}} = \mathbf{I}$.

c) The qubit is prepared in $|\psi(0)\rangle = |+\rangle$. Give $|\psi(t)\rangle$. Show that a computational-basis measurement gives $P(0) = P(1) = 1/2$ at every time t , but that a Hadamard-basis measurement gives $P(+) = \cos^2(\omega t/2)$ and $P(-) = \sin^2(\omega t/2)$.

d) (*for 7383 students only*) Find the earliest time $t > 0$ at which a Hadamard-basis measurement is certain to give the outcome associated with $|-\rangle$. Then state what the energy difference $E_1 - E_0$ has to do with that time.

6. (Introduction to classical information theory — 15 pts for 5383; 18 pts for 7383) This problem is self-contained and uses only the definitions given here. Consider a discrete memoryless source that emits symbols from an alphabet $\{x_1, x_2, \dots, x_n\}$ with probabilities $\{p_1, p_2, \dots, p_n\}$, where $\sum_j p_j = 1$. The **self-information** of the symbol x_j and the **Shannon entropy** of the source are defined as

$$I(x_j) = -\log_2 p_j \text{ bits}, \quad H(X) = \sum_{j=1}^n p_j I(x_j) = -\sum_{j=1}^n p_j \log_2 p_j \text{ bits}.$$

The entropy is the average number of bits of information carried by one symbol, and a term with $p_j = 0$ contributes zero. You may use the values $\log_2 3 \approx 1.585$, $\log_2 5 \approx 2.322$, and $\log_2 7 \approx 2.807$.

a) A source emits four symbols $\{A, B, C, D\}$ with probabilities $\{1/2, 1/4, 1/8, 1/8\}$. Compute the self-information of each symbol in bits and compute the entropy $H(X)$ of the source. Compare $H(X)$ with the 2 bits per symbol that a fixed-length binary code would require, and state what the comparison means.

b) Consider a binary source with probabilities $\{p, 1 - p\}$. Write $H(p)$ and evaluate it at $p = 0$, $p = 1/2$, and $p = 1$. State which value of p makes the entropy largest and which values make it zero, and explain both answers in terms of how uncertain the outcome is.

c) Return to the qubit state $|\psi\rangle$ of Problem 3. A computational-basis measurement produces outcomes 0 and 1 with the probabilities you found in Problem 3(b), and a Hadamard-basis measurement produces outcomes with the probabilities from Problem 3(c). Treating each measurement as a binary information source, compute the Shannon entropy of the outcome in each basis. Report both to three decimal places and comment on which measurement is less predictable.

d) (*for 7383 students only*) For the four-symbol source of part (a), construct a binary prefix code by assigning the codewords $A \rightarrow 0$, $B \rightarrow 10$, $C \rightarrow 110$, $D \rightarrow 111$. Verify that the code is a prefix code, compute its average codeword length in bits per symbol, and compare that length with $H(X)$. Then decode the received bit stream 100111110 and state why no separators between codewords are needed.