ECE 5/7383 Introduction to Quantum Informatics

Homework 4

Schrödinger's equation models the dynamic evolution of a wave function in a quantum system, and is given in terms of the Hamiltonian operator as:

$$i\hbar\frac{\partial}{\partial t}\left|\boldsymbol{\Psi}(t)\right\rangle = \hat{\mathbf{H}}\left|\boldsymbol{\Psi}(t)\right\rangle$$

Let the initial quantum state be

$$\left|\boldsymbol{\Psi}(0)\right\rangle = \left|0\right\rangle,$$

and the Hamiltonian be a scaled version of the Pauli-**X** operator,

$$\hat{\mathbf{H}} = \gamma\mathbf{X} = \gamma\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

a) Give an expression for $|\boldsymbol{\Psi}(t)\rangle$ in the computational, $|0\rangle$, $|1\rangle$.

b) Give an expression for $|\boldsymbol{\Psi}(t)\rangle$ in the alternative basis $|+\rangle$, $|-\rangle$, where

$$\left|+\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}, \left|-\right\rangle = \frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}.$$

c) Find the projector operator, **P**, to transform from the computational basis to the alternative basis, $|+\rangle$, $|-\rangle$, and show that when **P** is applied to the result of part a), you get the result of part b).

2. It is desired to construct an operator, **U**, that causes a time evolution of a qubit, $|\boldsymbol{\Psi}(t)\rangle$, as shown in the quantum circuit below.

$$\left|\Psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle \;\text{---}\boxed{\mathbf{U}}\text{---}\; \left|\Psi\right\rangle = \beta\left|0\right\rangle + \alpha\left|1\right\rangle$$

You only have two operators available, the Hadamard, **H**, and the Pauli-**Z** gates whose transfer matrices are:

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find the circuit comprised only of **H** and **Z** gates. You can use multiple instances of each gate, but you should use as few as possible for full credit. No credit for just using trial and error to find the right combination, you must show the theoretical basis you used to derive your answer. (Hint: Consider **U** to be a projection matrix and find its spectral decomposition.)