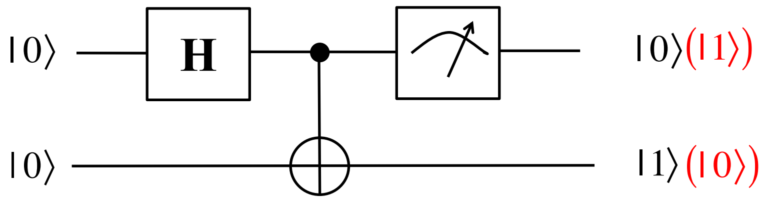
Big Picture Motivation



Representation, models, and the origin of Quantum Mechanics is essential to understand the Quantum Computing Program depicted above

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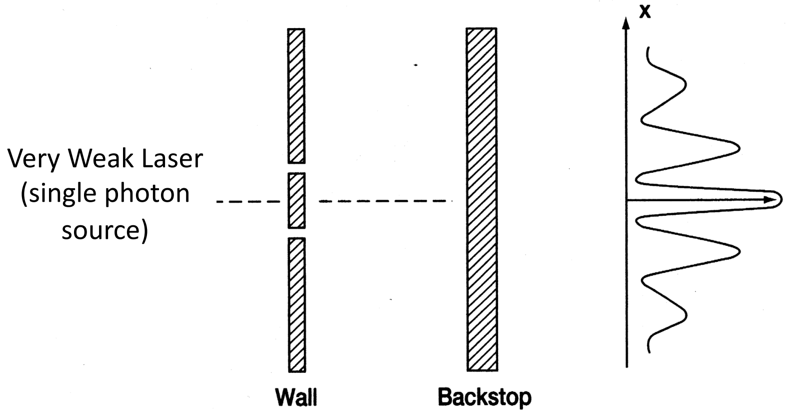
Young's Double Slit Experiment as a Computation

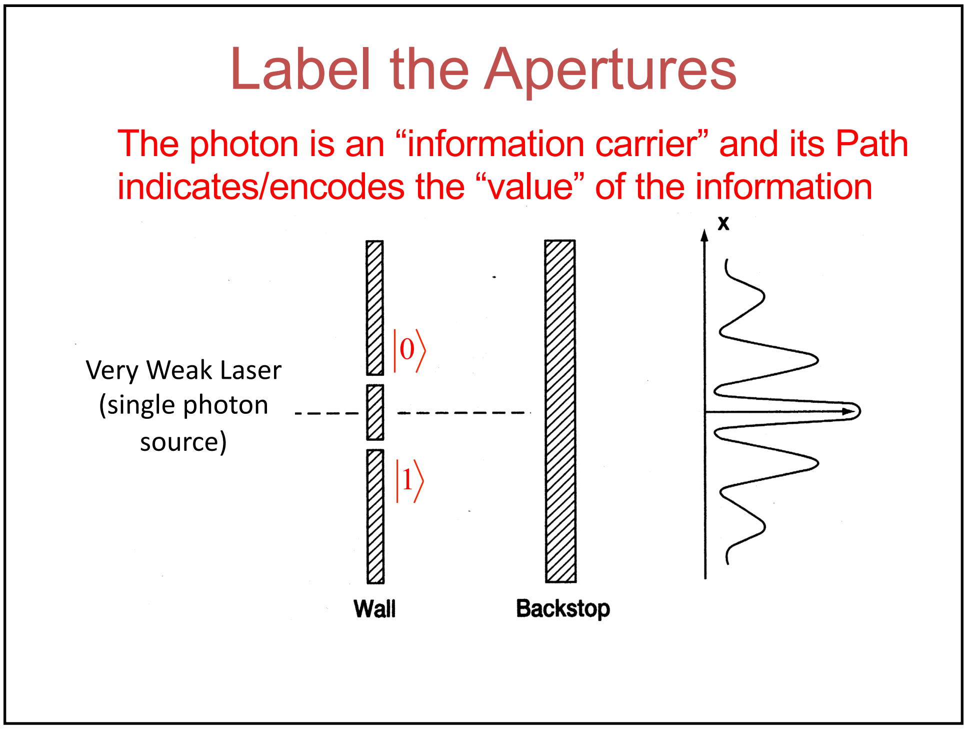

Very Weak Laser (single photon source)

Wall

Backstop

**A.D. Aczel, Entanglement, 2002, ISBN 1-55192-549-4*

2



Label the Apertures

The photon is an “information carrier” and its Path indicates/encodes the “value” of the information

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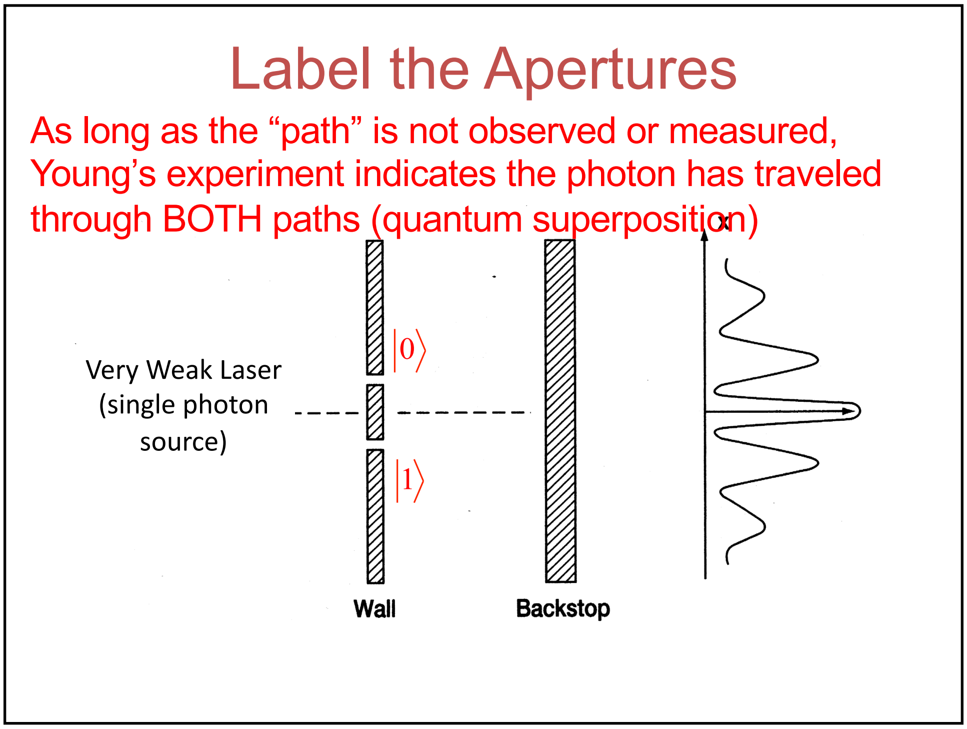

Label the Apertures

As long as the “path” is not observed or measured, Young’s experiment indicates the photon has traveled through BOTH paths (quantum superposition)

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Thomas Young's Experiment

"We choose to examine a phenomenon (the double-slit experiment) that is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality it contains the *only* mystery."

-Richard Feynman

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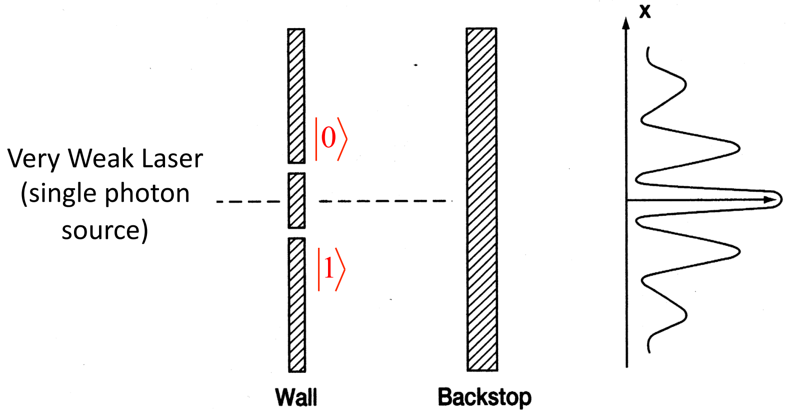
Label the Apertures

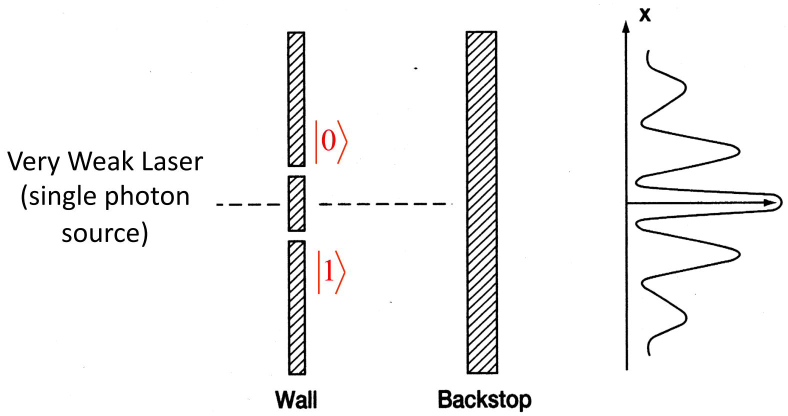
Thus, the photon is now carrying an information content of both "1" AND "0" !!!!! Referred to as $|0\rangle$ and $|1\rangle$.



6

Label the Apertures

Thus, the photon is now carrying an information content of both “1” AND “0” !!!!!



But, when we *measure* or *observe* the path, it will appear to only have traveled through either the top or the bottom path

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Double Slit Computation

- The CHARACTERISTIC/PROPERTY of the photon is serving as the Quantum Digit (or qubit)
  - in this case, the characteristic is its location (quantum observable)
- Causing the photon to have apparently traveled through BOTH paths is an example of “Quantum Superposition”
- Initializing qubits to be in a state of superposition is a common initial step in quantum informatics processing
- This is modeled mathematically as a “Hadamard” operation
- Denoted symbolically as:

$$|\psi\rangle \;\text{———}\; \boxed{\mathbf{H}} \;\text{———}\; |\varphi\rangle$$

$$|\varphi\rangle = \mathbf{H}|\psi\rangle$$

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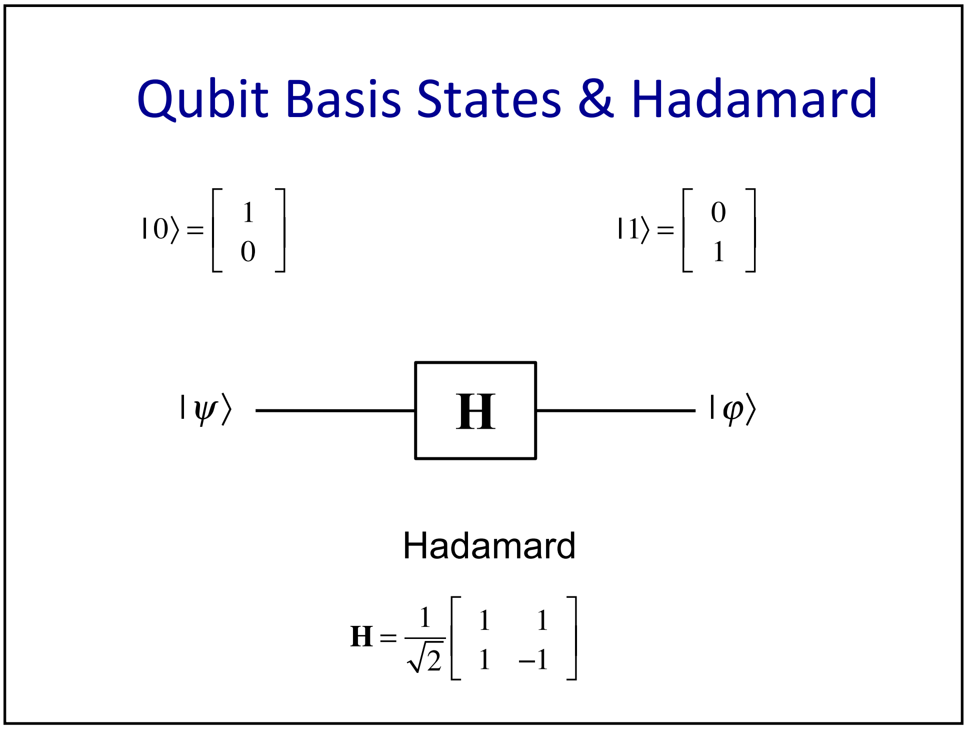

Qubit Basis States & Hadamard

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hadamard

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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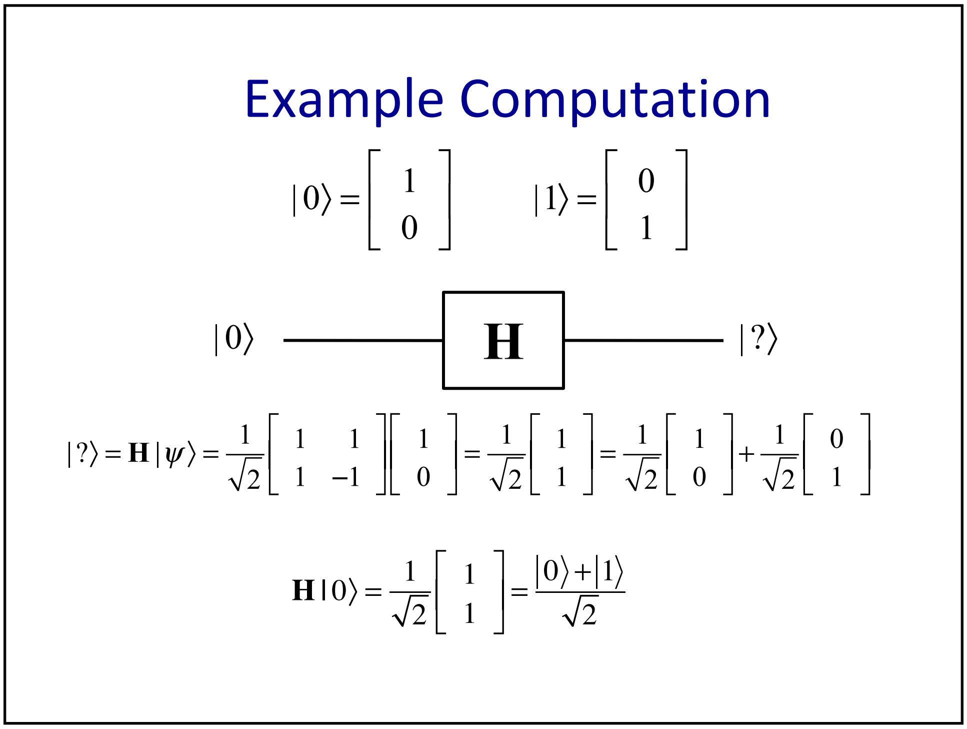

Example Computation

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|?\rangle = \mathbf{H}\,|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{H}\,|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

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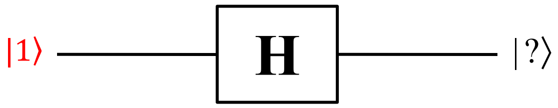
Example Computation (Init. to ket-1)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$|?\rangle = \mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

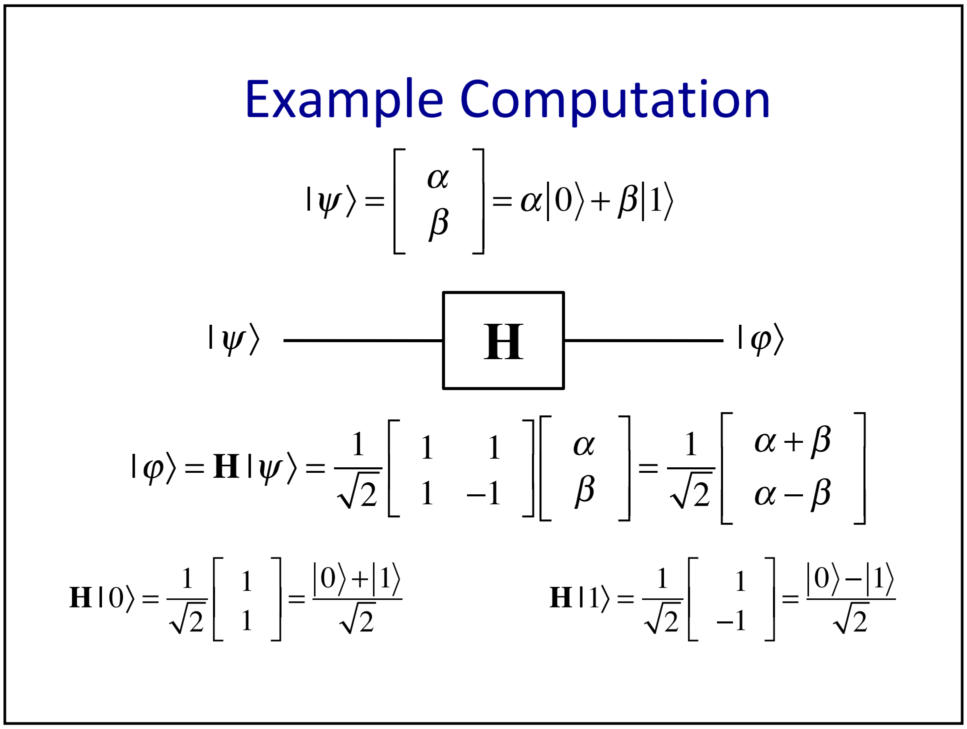
$$\mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Qubit Model

- Qubit exists in Linear Combination of Basis States
- “Ket" Notation Represents a Column Vector

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$



Example Computation

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle \;\text{———}\; \boxed{\mathbf{H}} \;\text{———}\; |\varphi\rangle$$

$$|\varphi\rangle = \mathbf{H}\,|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha+\beta \\ \alpha-\beta \end{bmatrix}$$

$$\mathbf{H}\,|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \qquad \mathbf{H}\,|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

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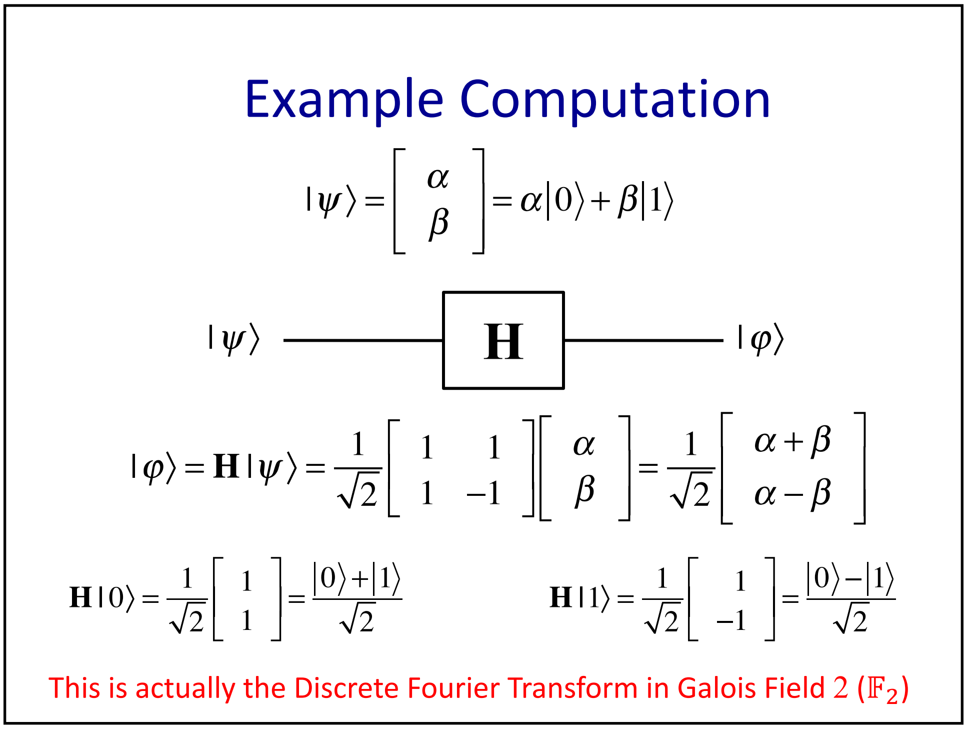

Example Computation

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This is actually the Discrete Fourier Transform in Galois Field 2 ($\mathbb{F}_2$)

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Physics/Math Class Content

- This is why we are reviewing classical and quantum behavior
- It is also why we are reviewing linear algebra
- QI is based on QM theory
- QI theory is described with linear algebra
- Please have patience, after one or two more classes, we will be prepared to focus solely on Quantum Informatics !!!!!

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Anyone who is not shocked about quantum theory has not understood it
Niels Bohr

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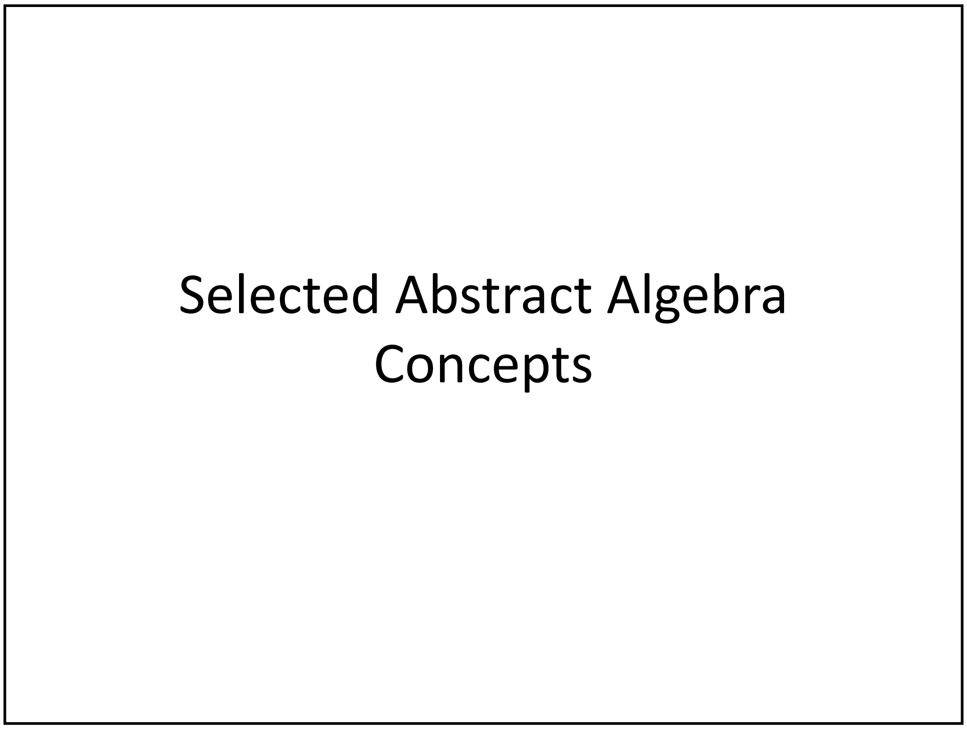

Selected Abstract Algebra Concepts

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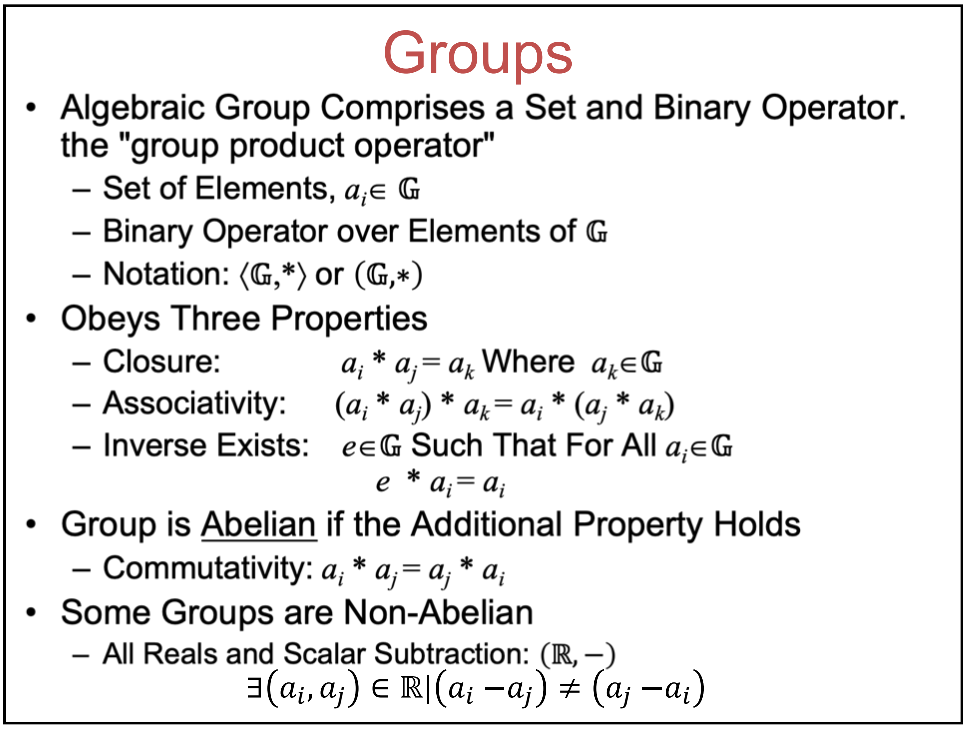

Groups

- Algebraic Group Comprises a Set and Binary Operator. the "group product operator"
  - Set of Elements, $a_i \in \mathbb{G}$
  - Binary Operator over Elements of $\mathbb{G}$
  - Notation: $\langle \mathbb{G}, * \rangle$ or $(\mathbb{G}, *)$
- Obeys Three Properties
  - Closure: $a_i * a_j = a_k$ Where $a_k \in \mathbb{G}$
  - Associativity: $(a_i * a_j) * a_k = a_i * (a_j * a_k)$
  - Inverse Exists: $e \in \mathbb{G}$ Such That For All $a_i \in \mathbb{G}$

    $e * a_i = a_i$
- Group is <u>Abelian</u> if the Additional Property Holds
  - Commutativity: $a_i * a_j = a_j * a_i$
- Some Groups are Non-Abelian
  - All Reals and Scalar Subtraction: $(\mathbb{R}, -)$

    $\exists (a_i, a_j) \in \mathbb{R} | (a_i - a_j) \neq (a_j - a_i)$

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Group Axioms using Quantifiers

- A Group is an Algebraic Structure Composed of a Set of elements with an Associated Binary Operator usually called Multiplication or the Group Product Operator

group notation

$(\mathbb{G},*)$

binary product operator

$*: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$

- A Group Must Satisfy the Three "Group Axioms" with closure:

G1: Associativity with respect to the group operator:

$$\forall (a, b, c) \in \mathbb{G}, a * (b * c) = (a * b) * c$$

G2: Identity Element Exists with respect to group operator:

$$\exists e_i \in \mathbb{G} \mid a_i * e_i = e_i * a_i = a_i, \forall a_i \in \mathbb{G}$$

G3: Inverse Elements Exist:

$$\forall a_i \in \mathbb{G},\ \exists a_i^{-1} \in \mathbb{G} \mid a_i * a_i^{-1} = a_i^{-1} * a_i = e_i$$

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Abelian Group Axiom

- A Group that Also Obeys the Property of Commutativity is a Commutative or Abelian Group:

group notation

$(\mathbb{G},*)$

binary product operator

$*: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$

G4: Commutativity (not required for group to exist):

$$\forall (a_i, a_j) \in \mathbb{G}, a_i * a_j = a_j * a_i$$

- If Commutativity is not Obeyed, the Group is said to be non-Abelian or non-Commutative

Niels Henrik Abel

Proved that there is no general solution of a fifth-degree polynomial using radicals. Performed most of his research in poverty and died of tuberculosis at age 26.

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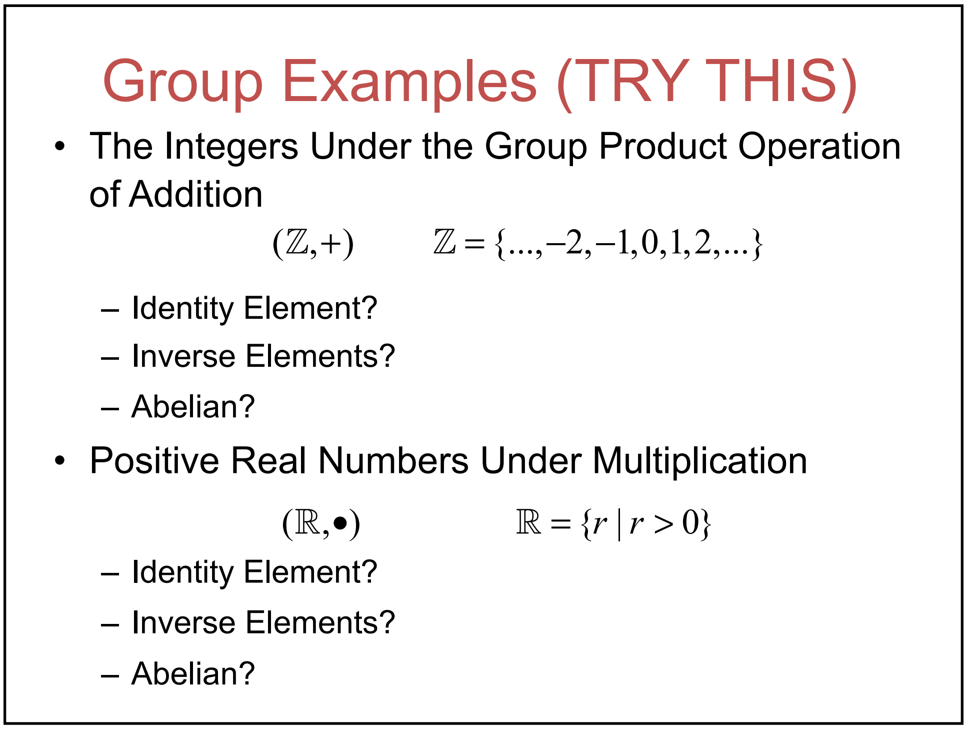

Group Examples (TRY THIS)

- The Integers Under the Group Product Operation of Addition

$$(\mathbb{Z},+) \qquad \mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$$

  - Identity Element?
  - Inverse Elements?
  - Abelian?
- Positive Real Numbers Under Multiplication

$$(\mathbb{R},\bullet) \qquad \mathbb{R} = \{r \mid r > 0\}$$

  - Identity Element?
  - Inverse Elements?
  - Abelian?

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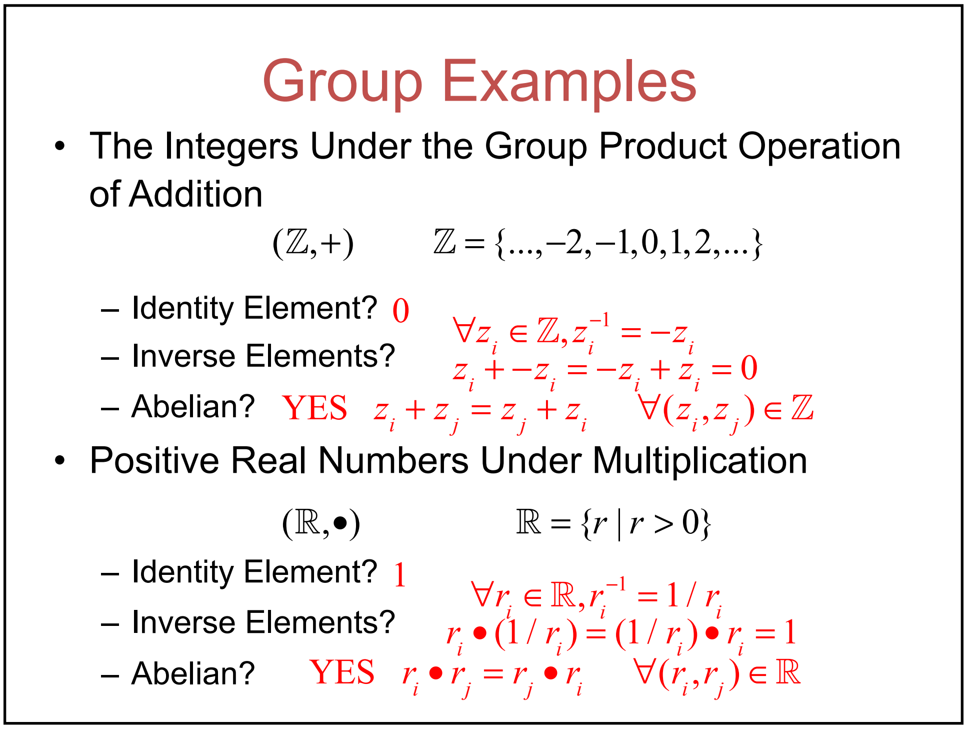

Group Examples

- The Integers Under the Group Product Operation of Addition

$$(\mathbb{Z},+) \qquad \mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$$

  - Identity Element? 0
  - Inverse Elements? $\forall z_i \in \mathbb{Z}, z_i^{-1} = -z_i$ $\quad z_i + -z_i = -z_i + z_i = 0$
  - Abelian? YES $z_i + z_j = z_j + z_i \quad \forall (z_i, z_j) \in \mathbb{Z}$
- Positive Real Numbers Under Multiplication

$$(\mathbb{R},\bullet) \qquad \mathbb{R} = \{r \mid r > 0\}$$

  - Identity Element? 1
  - Inverse Elements? $\forall r_i \in \mathbb{R}, r_i^{-1} = 1/r_i$ $\quad r_i \bullet (1/r_i) = (1/r_i) \bullet r_i = 1$
  - Abelian? YES $r_i \bullet r_j = r_j \bullet r_i \quad \forall (r_i, r_j) \in \mathbb{R}$

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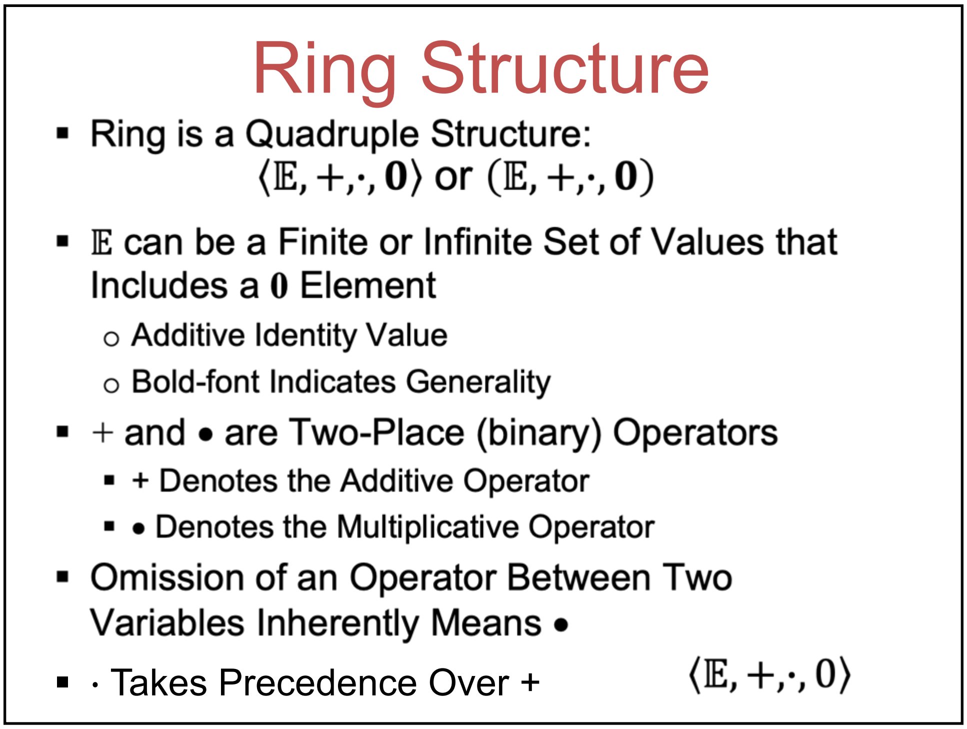

Ring Structure

- Ring is a Quadruple Structure:
  $\langle \mathbb{E}, +, \cdot, \mathbf{0} \rangle$ or $(\mathbb{E}, +, \cdot, \mathbf{0})$
- $\mathbb{E}$ can be a Finite or Infinite Set of Values that Includes a $\mathbf{0}$ Element
  - o Additive Identity Value
  - o Bold-font Indicates Generality
- $+$ and • are Two-Place (binary) Operators
  - + Denotes the Additive Operator
  - • Denotes the Multiplicative Operator
- Omission of an Operator Between Two Variables Inherently Means •
- · Takes Precedence Over +

$\langle \mathbb{E}, +, \cdot, 0 \rangle$

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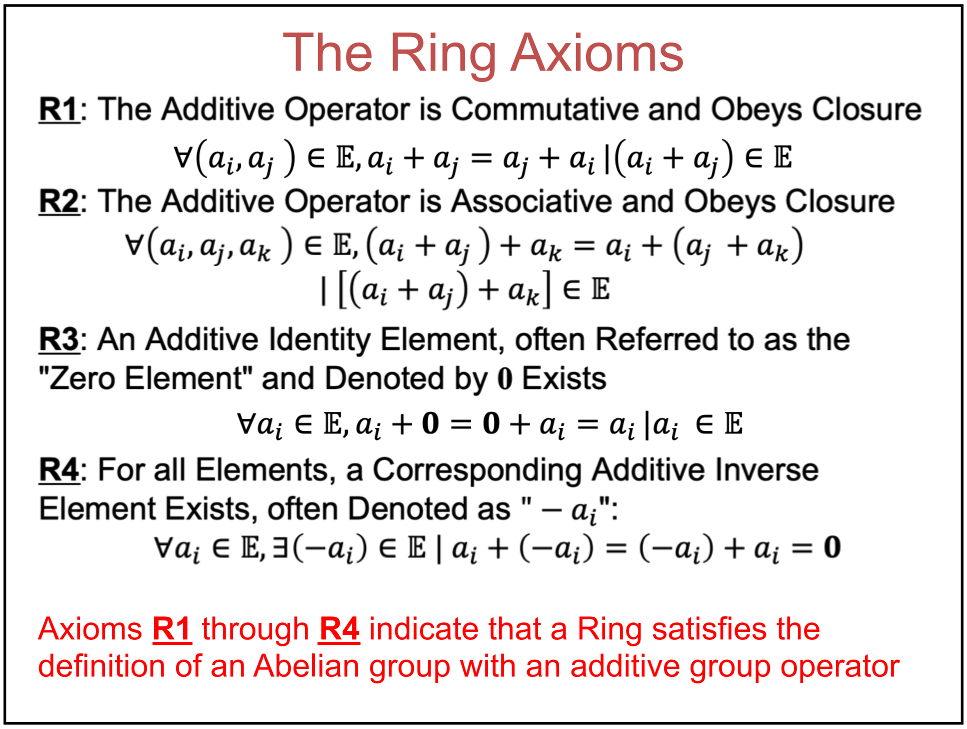

The Ring Axioms

R1: The Additive Operator is Commutative and Obeys Closure

$$\forall(a_i, a_j) \in \mathbb{E}, a_i + a_j = a_j + a_i \,|\, (a_i + a_j) \in \mathbb{E}$$

R2: The Additive Operator is Associative and Obeys Closure

$$\forall(a_i, a_j, a_k) \in \mathbb{E}, (a_i + a_j) + a_k = a_i + (a_j + a_k)$$
$$|\, \left[(a_i + a_j) + a_k\right] \in \mathbb{E}$$

R3: An Additive Identity Element, often Referred to as the "Zero Element" and Denoted by $\mathbf{0}$ Exists

$$\forall a_i \in \mathbb{E}, a_i + \mathbf{0} = \mathbf{0} + a_i = a_i \,|\, a_i \in \mathbb{E}$$

R4: For all Elements, a Corresponding Additive Inverse Element Exists, often Denoted as "$-a_i$":

$$\forall a_i \in \mathbb{E}, \exists(-a_i) \in \mathbb{E} \,|\, a_i + (-a_i) = (-a_i) + a_i = \mathbf{0}$$

Axioms **R1** through **R4** indicate that a Ring satisfies the definition of an Abelian group with an additive group operator

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The Ring Axioms (cont.)

R5: The Multiplicative Operator is Associative and Obeys Closure

$$\forall(a_i, a_j, a_k) \in \mathbb{E}, (a_i \cdot a_j) \cdot a_k = a_i \cdot (a_j \cdot a_k) \mid [(a_i \cdot a_j) \cdot a_k] \in \mathbb{E}$$

R6: Distributivity over the Additive Operator with Closure

$$\forall(a_i, a_j, a_k) \in \mathbb{E}, a_i + (a_j \cdot a_k) = (a_i + a_j) \cdot (a_i + a_k) \mid [a_i + (a_j \cdot a_k)] \in \mathbb{E}$$

R7: Distributivity over the Multiplicative Operator with Closure

$$\forall(a_i, a_j, a_k) \in \mathbb{E}, a_i \cdot (a_j + a_k) = (a_i \cdot a_j) + (a_i \cdot a_k) \mid [a_i \cdot (a_j + a_k)] \in \mathbb{E}$$

A Ring is further referred to as a **Commutative Ring** when:

$$\forall(a_i, a_j) \in \mathbb{E}, a_i \cdot a_j = a_j \cdot a_i \mid (a_i \cdot a_j) \in \mathbb{E}$$

A Ring is further referred to as **Unital** or a **Ring with Identity** when it comprises a Multiplicative Identity Element, $\mathbf{1}$:

$$\forall a_i \in \mathbb{E}, a_i \cdot \mathbf{1} = \mathbf{1} \cdot a_i = a_i \mid a_i \in \mathbb{E}$$

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Field Structure

- A Field $\mathbb{F}$ is an algebraic structure with two associated binary operators usually referred to as addition and multiplication
- A Field also Obeys the following Three Axioms:
  1. Under Addition, $\mathbb{F}$ is an **Abelian Group** with Multiplicative Identity Element $\mathbf{0}$ Such That:

     $$\mathbf{0} + a_i = a_i \forall a_i \in \mathbb{F}$$

  2. Under Multiplication, the non-zero elements of $\mathbb{F}$ form an Abelian Group with Identity Element 1 Such That:

     $$\forall a_i \in \mathbb{F}, \mathbf{1} \cdot a_i = a_i \qquad \forall a_i \in \mathbb{F}, \mathbf{0} \cdot a_i = \mathbf{0}$$

  3. Distributivity Over the Multiplicative Operator Holds:

     $$\forall(a_i, a_j, a_k) \in \mathbb{F}, a_i \cdot (a_j + a_k) = (a_i \cdot a_j) + (a_i \cdot a_k)$$

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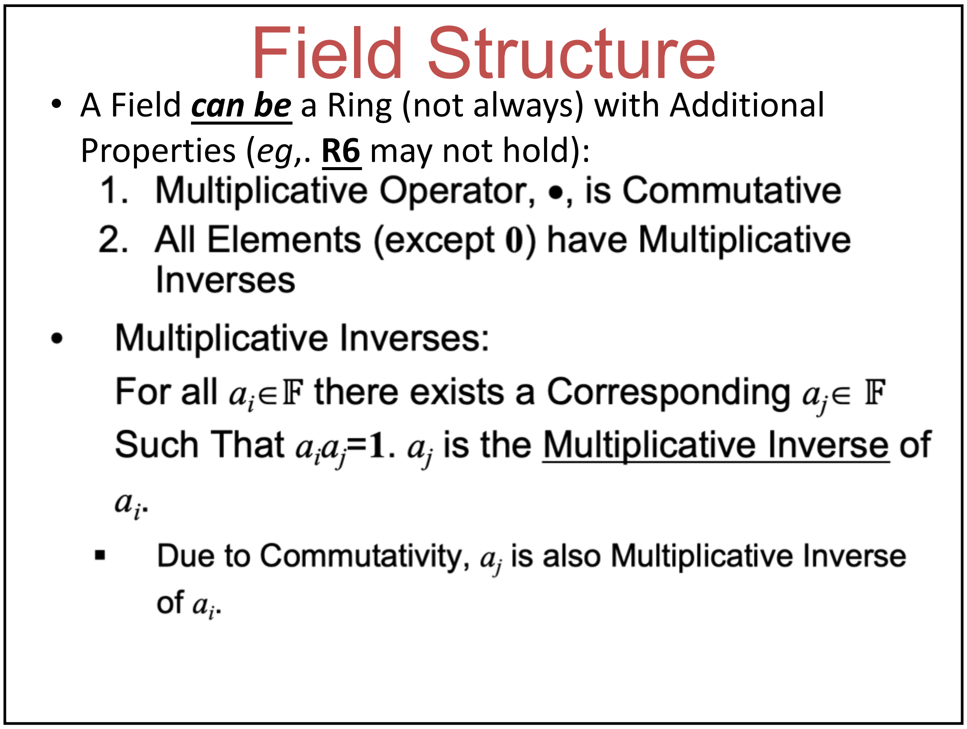

Field Structure

- A Field ***can be*** a Ring (not always) with Additional Properties (*eg*,. **R6** may not hold):
  1. Multiplicative Operator, •, is Commutative
  2. All Elements (except $\mathbf{0}$) have Multiplicative Inverses
- Multiplicative Inverses:
  For all $a_i \in \mathbb{F}$ there exists a Corresponding $a_j \in \mathbb{F}$ Such That $a_i a_j = \mathbf{1}$. a_j is the Multiplicative Inverse of a_i.
  - Due to Commutativity, a_j is also Multiplicative Inverse of a_i.

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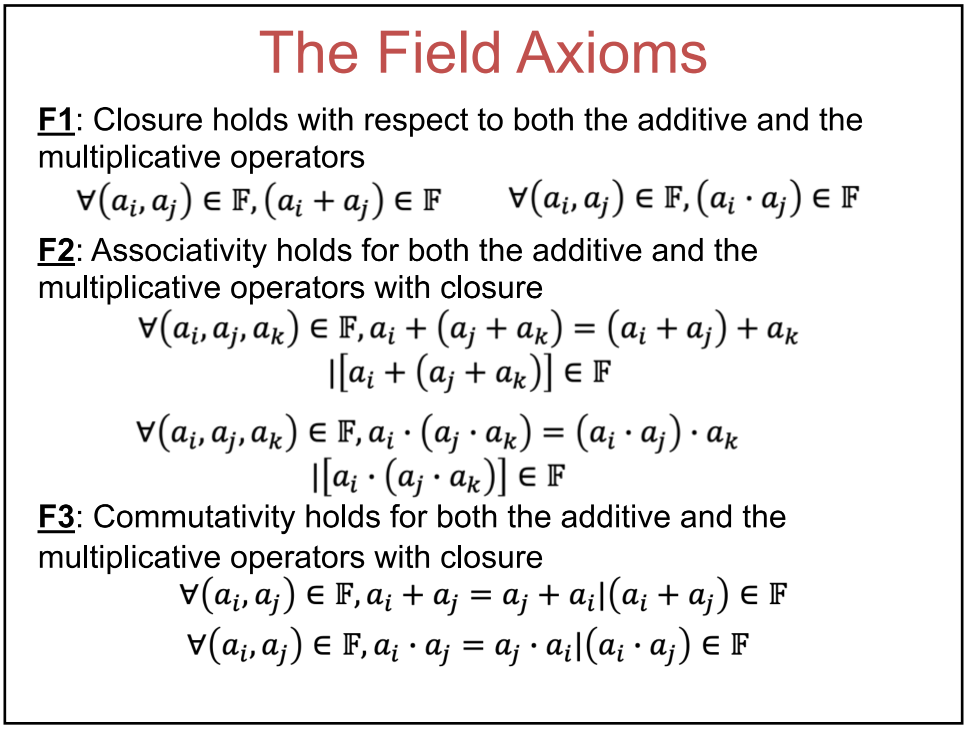

The Field Axioms

F1: Closure holds with respect to both the additive and the multiplicative operators

$$\forall(a_i, a_j) \in \mathbb{F}, (a_i + a_j) \in \mathbb{F} \qquad \forall(a_i, a_j) \in \mathbb{F}, (a_i \cdot a_j) \in \mathbb{F}$$

F2: Associativity holds for both the additive and the multiplicative operators with closure

$$\forall(a_i, a_j, a_k) \in \mathbb{F}, a_i + (a_j + a_k) = (a_i + a_j) + a_k$$
$$|[a_i + (a_j + a_k)] \in \mathbb{F}$$

$$\forall(a_i, a_j, a_k) \in \mathbb{F}, a_i \cdot (a_j \cdot a_k) = (a_i \cdot a_j) \cdot a_k$$
$$|[a_i \cdot (a_j \cdot a_k)] \in \mathbb{F}$$

F3: Commutativity holds for both the additive and the multiplicative operators with closure

$$\forall(a_i, a_j) \in \mathbb{F}, a_i + a_j = a_j + a_i | (a_i + a_j) \in \mathbb{F}$$
$$\forall(a_i, a_j) \in \mathbb{F}, a_i \cdot a_j = a_j \cdot a_i | (a_i \cdot a_j) \in \mathbb{F}$$

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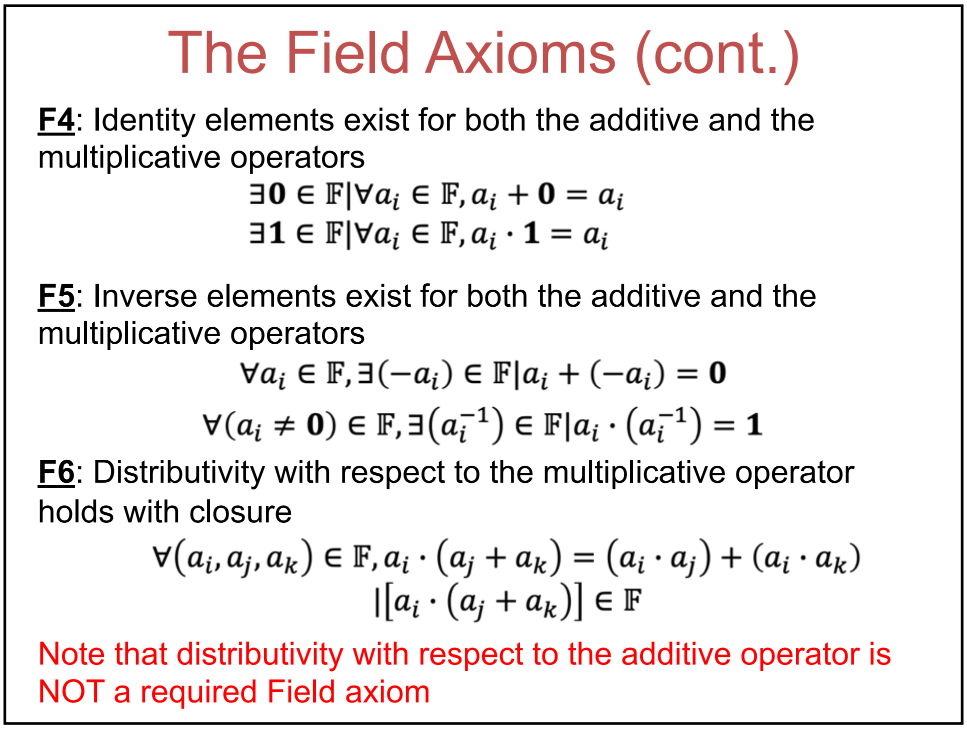

The Field Axioms (cont.)

F4: Identity elements exist for both the additive and the multiplicative operators

$$\exists \mathbf{0} \in \mathbb{F} | \forall a_i \in \mathbb{F}, a_i + \mathbf{0} = a_i$$
$$\exists \mathbf{1} \in \mathbb{F} | \forall a_i \in \mathbb{F}, a_i \cdot \mathbf{1} = a_i$$

F5: Inverse elements exist for both the additive and the multiplicative operators

$$\forall a_i \in \mathbb{F}, \exists(-a_i) \in \mathbb{F} | a_i + (-a_i) = \mathbf{0}$$
$$\forall(a_i \neq \mathbf{0}) \in \mathbb{F}, \exists\left(a_i^{-1}\right) \in \mathbb{F} | a_i \cdot \left(a_i^{-1}\right) = \mathbf{1}$$

F6: Distributivity with respect to the multiplicative operator holds with closure

$$\forall\left(a_i, a_j, a_k\right) \in \mathbb{F}, a_i \cdot \left(a_j + a_k\right) = \left(a_i \cdot a_j\right) + \left(a_i \cdot a_k\right)$$
$$|\left[a_i \cdot \left(a_j + a_k\right)\right] \in \mathbb{F}$$

Note that distributivity with respect to the additive operator is NOT a required Field axiom

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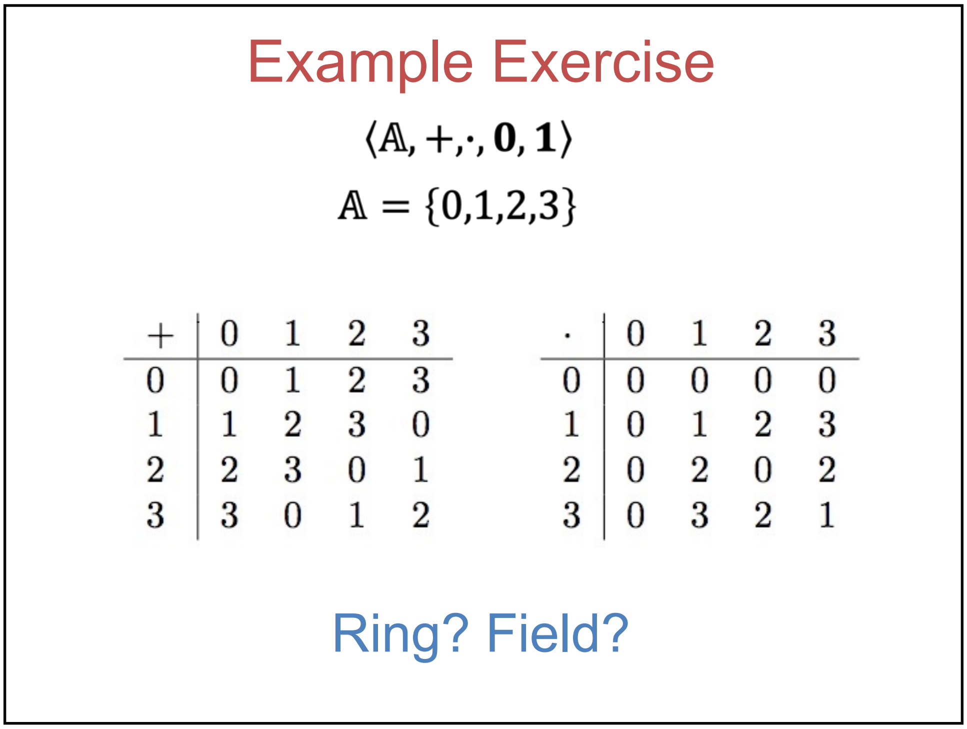

Example Exercise

$$\langle \mathbb{A}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$$

$$\mathbb{A} = \{0,1,2,3\}$$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Ring? Field?

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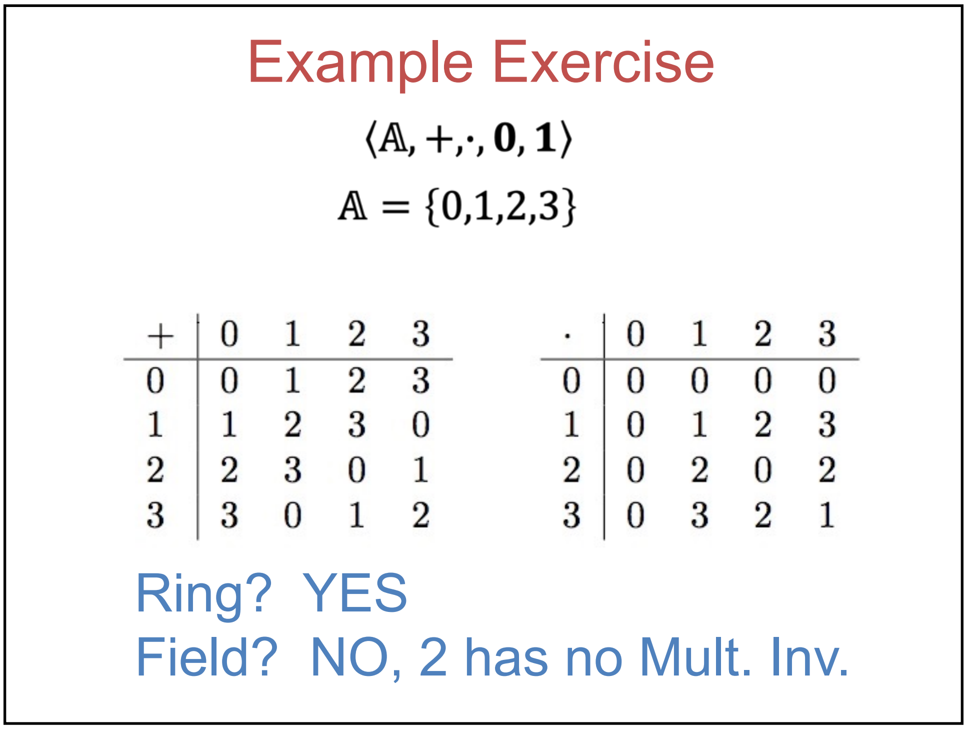


Example Exercise

$\langle \mathbb{A}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$

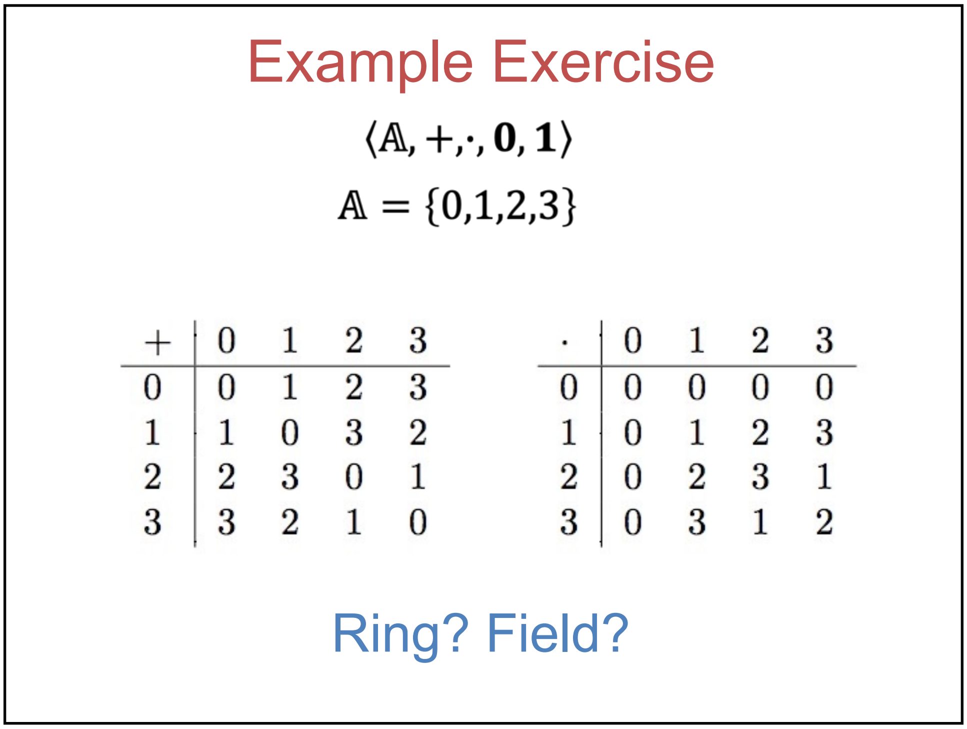
$\mathbb{A} = \{0,1,2,3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Ring? YES

Field? NO, 2 has no Mult. Inv.



Example Exercise

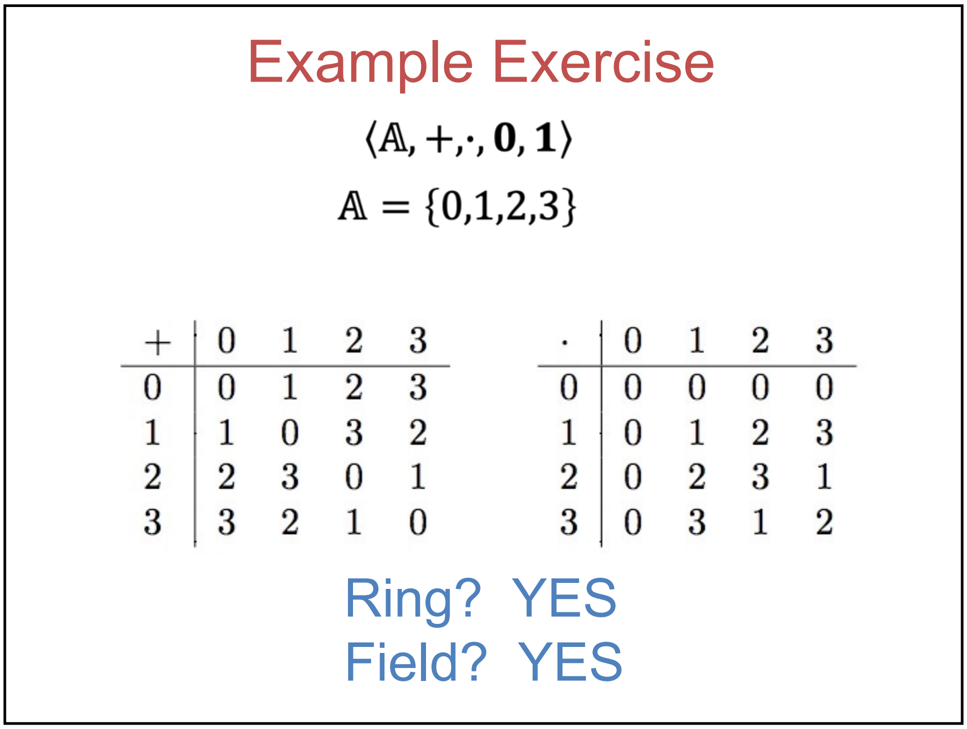
$\langle \mathbb{A}, +, \cdot, \mathbf{0}, \mathbf{1} \rangle$

$\mathbb{A} = \{0,1,2,3\}$

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Ring? Field?



Example Exercise

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·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Ring? YES

Field? YES

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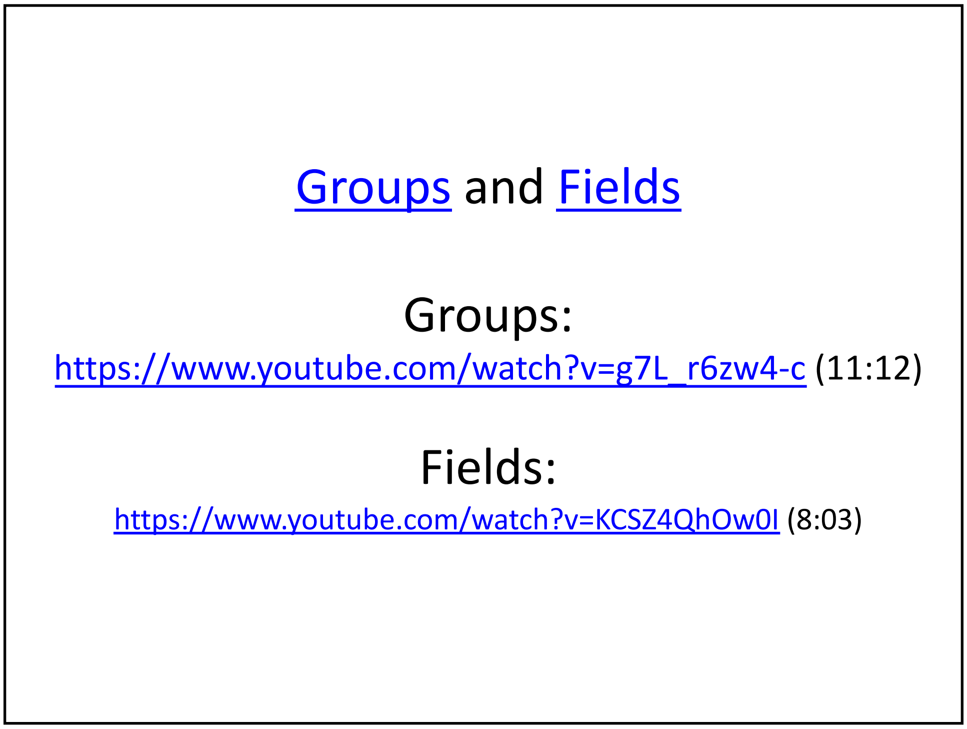

Groups and Fields

Groups:

https://www.youtube.com/watch?v=g7L_r6zw4-c (11:12)

Fields:

https://www.youtube.com/watch?v=KCSZ4QhOw0I (8:03)

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Algebra

- Not as Strictly Defined as Groups, Rings, Fields
- General Definition is a Structure Comprised of:
  - At least one set of Elements, $\mathbb{A}$, (can be more than one)
  - One or Axioms over the one or more sets of elements
- Another definition is a field of study of "algebraic structures"
- EXAMPLES
  - The Field of Reals
  - The Set $\mathbb{B}$ with the Huntington Postulates (Switching Algebra)
  - The Set $\{0,1,2\}$ with $min\{x,y\}$, $max\{x,y\}$, $J_0(x)$,$J_1(x)$,$J_2(x)$(3-valued Post Algebra) where

$$J_0(x) = \begin{cases} 2, x = 0 \\ 0, \text{otherwise} \end{cases} \quad J_1(x) = \begin{cases} 2, x = 1 \\ 0, \text{otherwise} \end{cases} \quad J_2(x) = \begin{cases} 2, x = 2 \\ 0, \text{otherwise} \end{cases}$$

Lattice Algebra

Lattice Algebra – defined by the tuple:

$$\langle A, \vee, \wedge \rangle$$

Where:

A is a non-empty set

$\vee, \wedge$ are binary operations disjunction and conjunction

And, the Following Axioms Hold:

$$\begin{array}{lll} a \vee a = a & a \wedge a = a & \text{(Idempotence)} \\ a \vee b = b \vee a & a \wedge b = b \wedge a & \text{(Commutativity)} \\ a \vee (b \vee c) = (a \vee b) \vee c & a \wedge (b \wedge c) = (a \wedge b) \wedge c & \text{(Associativity)} \\ a \vee (a \wedge b) = a & a \wedge (a \vee b) = a & \text{(Absorption)} \\ & a,b,c \in A & \end{array}$$

Distributive Lattice Algebra

Distributive Lattice Algebra

And, the Following Distributive Laws Hold:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Complemented Distributive Lattice Algebra

1) maximal element = 1
2) minimal element = 0
3) For any $a \in A$ if $\exists\, x_a \in A$ such that $a \wedge x_a = 0$
4) For any $a \in A$ if $\exists\, x_a \in A$ such that $a \vee x_a = 1$

A Complemented Distributive Algebra is a Boolean Algebra

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Lattice Algebra

Lattice Algebra – defined by the tuple:

$\langle A, \vee, \bullet \rangle$

Where:

A is a non-empty set
$\vee, \bullet$ are binary operations

And, the Following Axioms Hold:

$a \vee a = a$	$a \bullet a = a$	(Idempotence)
$a \vee b = b \vee a$	$a \bullet b = b \bullet a$	(Commutativity)
$a \vee (b \vee c) = (a \vee b) \vee c$	$a \bullet (b \bullet c) = (a \bullet b) \bullet c$	(Associativity)
$a \vee (a \bullet b) = a$	$a \bullet (a \vee b) = a$	(Absorption)

$a,b,c \in A$

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Distributive Lattice Algebra

Distributive Lattice Algebra

And, the Following Distributive Laws Hold:

$$a \vee (b \bullet c) = (a \vee b) \bullet (a \vee c)$$
$$a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$$

Complemented Distributive Lattice Algebra

1) maximal element = 1
2) minimal element = 0
3) For any $a \in A$ if $\exists\ x_a \in A$ such that $a \bullet x_a = 0$
4) For any $a \in A$ if $\exists\ x_a \in A$ such that $a \vee x_a = 1$

A Complemented Distributive Algebra is a Boolean Algebra

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Boolean Algebra

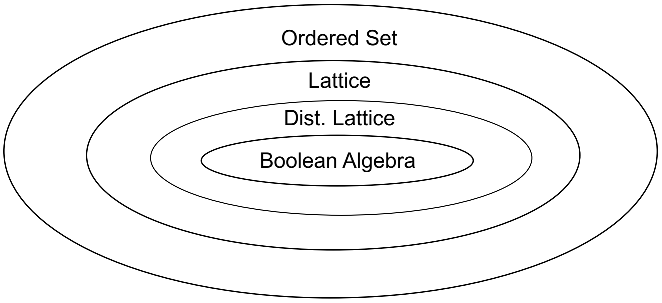
$$\langle B, \vee, \bullet, {}^{-}, 0, 1\rangle$$

$$0, 1 \in B$$

is $^{-}$ a unary operation over B

$\vee$, $\bullet$ are binary operations over B

0 is the “identity element” wrt $\vee$
1 is the “identity element” wrt $\bullet$



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Selected Linear Algebra Concepts

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Vector Space

- Consists of
  1) An Abelian (commutative) group $(\mathbb{V}, +)$ whose elements are called vectors and whose product operator is vector addition, characterized by a "dimension," n
  2) A field $\mathbb{F}$ (usually the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) whose elements are called "scalars"
  3) An multiplicative operation called "scaling" denoted by an absence of an operation symbol between a scalar and a vector that associates a scalar $\alpha \in \mathbb{F}$ and vector $\mathbf{x} \in \mathbb{V}$ and results in another vector $\alpha\mathbf{x} \in \mathbb{V}$ or $\alpha\mathbf{x} \in \mathbb{V}$, $\{\alpha, \mathbf{x}\} \rightarrow \alpha\mathbf{x}$

$$\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$$

$$(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$$

$$(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$$

$$(1)\mathbf{x} = \mathbf{x}$$

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Complex/Hilbert Vector Spaces

- Hilbert Space is generally an infinite-dimensional vector space
  - with an inner product
  - and with associated norm
- Quantum Computing Literature Traditionally Refers to Finite n-dimensional Complex Euclidean Vector Space as a Hilbert Space (technically correct)
- FOR OUR PURPOSES: Hilbert Space:
  n-dimensional vector space over the field of complex numbers with an inner product and associated norm

Vector Space Inner Products

- Consider an n-dimensional Vector Space:
- If, for all pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, an associated real number exists, an inner product $(\mathbf{x},\ \mathbf{y})$, such that the following conditions are satisfied:

$$(\mathbf{x},\mathbf{y}) = (\mathbf{y},\mathbf{x})$$

$$(c\mathbf{x},\mathbf{y}) = c(\mathbf{x},\mathbf{y}) \text{ if } c \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}$$

$$(\mathbf{x}+\mathbf{z},\mathbf{y}) = (\mathbf{x},\mathbf{y}) + (\mathbf{z},\mathbf{y}) \quad \forall \mathbf{z} \in \mathbb{R}^n \text{ (or } \mathbb{C}^n\text{)}$$

$$(\mathbf{x},\mathbf{x}) \geq 0 \text{ such that } \quad (\mathbf{x},\mathbf{x}) = 0 \text{ if and only if } \mathbf{x} = \mathbf{0}$$

- Then, we have an n-dimensional ***Euclidean*** Vector Space
- $(\mathbf{x},\ \mathbf{y})$ is the ***Inner Product*** of Vectors $\mathbf{x}$ and $\mathbf{y}$
  - "dot" product, $(\mathbf{x},\ \mathbf{y})=\mathbf{x}\bullet\mathbf{y}$, is one form of an inner product that yields a scalar product value
  - most common is the "Euclidean" inner product

Euclidean Vector Spaces

- Euclidean Spaces use “dot product” & “L_2 norm”
- “Length” of a Euclidean Vector & inner/dot product:

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})} \qquad (\mathbf{x},\mathbf{y}) = \mathbf{x}\bullet\mathbf{y}$$

- Euclidean Angle between the two vectors α and β:

$$\boldsymbol{\theta} = \cos^{-1}\frac{(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|\ \|\mathbf{y}\|} \qquad \cos(\boldsymbol{\theta}) = \frac{(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|\ \|\mathbf{y}\|}$$

- If $(\mathbf{x}, \mathbf{y})=0$, then $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal*
  - Orthogonality occurs when the inner product is zero
- From a Euclidean viewpoint, 90 degrees apart

$$\theta = \pi / 2 = 90^\circ$$

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Orthogonal Basis Sets

- Consider a set of *n* Vectors: $V = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$
- This set forms an ***Orthogonal Basis*** of the n-Dimensional Vector Space if: $(\mathbf{e}_i, \mathbf{e}_j) = 0, \forall i \neq j$
- Vector space elements (all vectors in the space) can ALWAYS be represented as a linear combination of scaled basis vectors, the ***Basis Set*** or Basis of the Space
  - Different/Alternative Basis sets (Bases) may be used to represent the SAME Vector Space
- Finding an alternative Basis Set is known as a ***Change of Basis*** and can be accomplished via a ***Basis Transform***
  - very important in QM, provides a different “point of view” of the Vector Space

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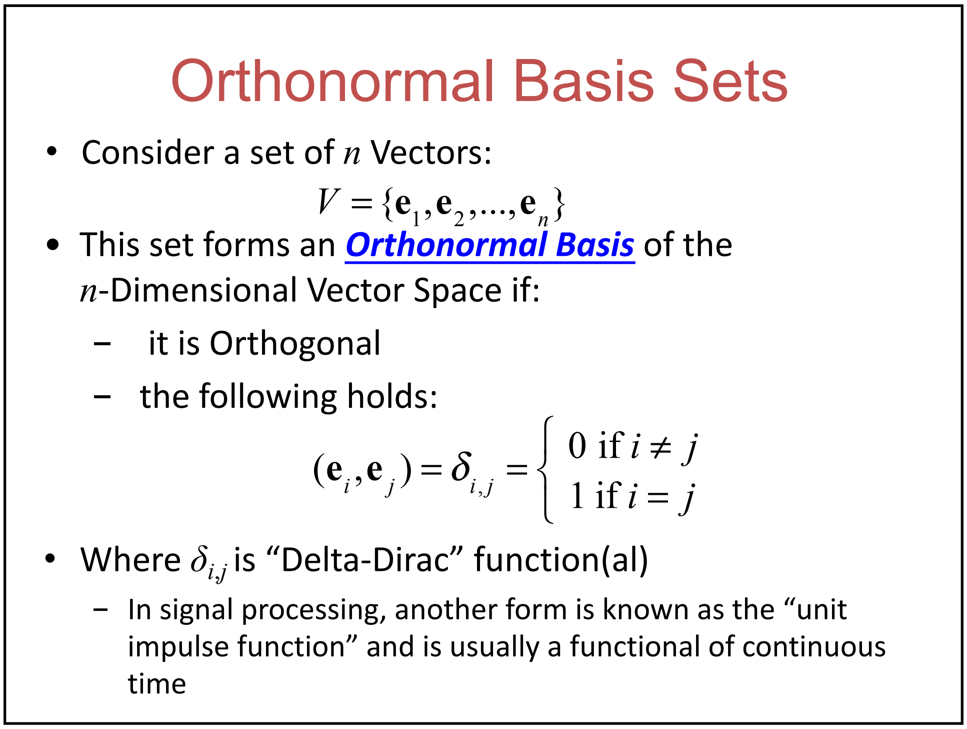

Orthonormal Basis Sets

- Consider a set of n Vectors:

$$V = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

- This set forms an ***Orthonormal Basis*** of the n-Dimensional Vector Space if:
  - it is Orthogonal
  - the following holds:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

- Where $\delta_{i,j}$ is "Delta-Dirac" function(al)
  - In signal processing, another form is known as the "unit impulse function" and is usually a functional of continuous time

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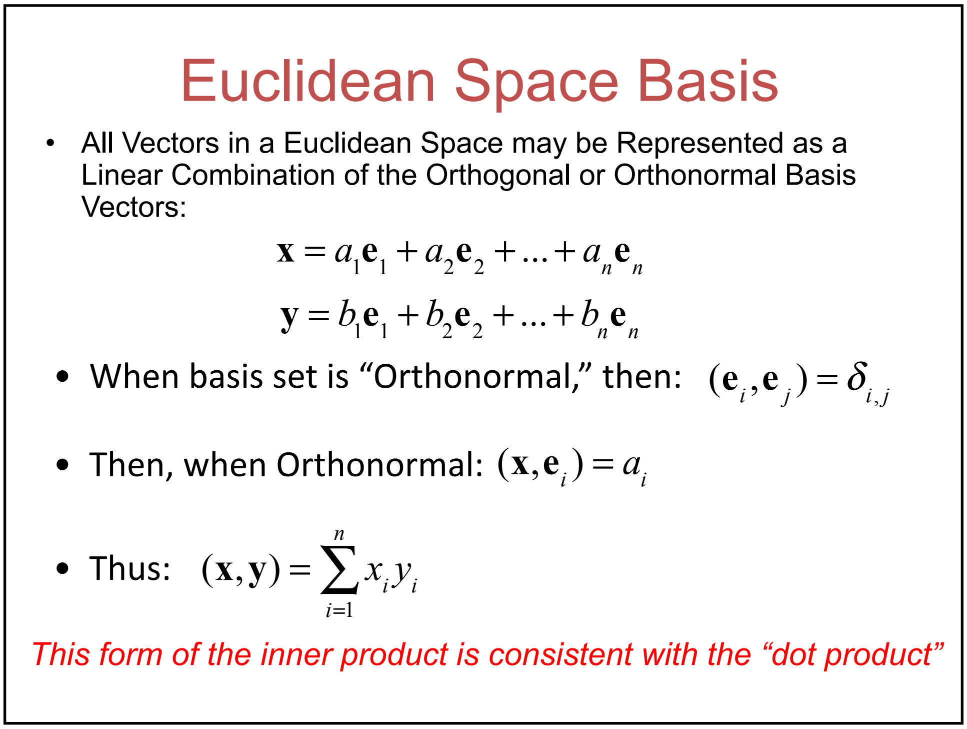

Euclidean Space Basis

- All Vectors in a Euclidean Space may be Represented as a Linear Combination of the Orthogonal or Orthonormal Basis Vectors:

$$\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_n\mathbf{e}_n$$

$$\mathbf{y} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \ldots + b_n\mathbf{e}_n$$

- When basis set is "Orthonormal," then: $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}$
- Then, when Orthonormal: $(\mathbf{x}, \mathbf{e}_i) = a_i$
- Thus: $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i$

This form of the inner product is consistent with the "dot product"

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Vector Spaces

https://www.youtube.com/watch?v=ozwodzD5bJM (6:57)

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Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \mathbf{A} = \left[a_{ij} \right]_{n \times m}$$

- $\mathbf{A}$ maps/transforms/operates on Vectors from a Vector Space of Dimension n to Vector Space of Dimension m: $\mathbf{A} : \mathbb{V}^n \to \mathbb{V}^m$
- When $\mathbf{A}$ is a Square Matrix, n=m, it Represents a Linear Mapping to od vector to another within the same space: $\mathbf{A} : \mathbb{V}^n \to \mathbb{V}^n$
- Each Row of $\mathbf{A}$ is a Row Vector and Each Column is a Column Vector
- Row/Column Vectors Span the Domain/Range of the Vector Spaces, $\mathbb{V}^n$ and $\mathbb{V}^m$

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Elementary Row Operations

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

1. Any row may be interchanged with any other
2. Any row may be replaced by itself multiplied by a constant
3. Any row may be replaced by the column-wise sum of itself and a multiple of another row

Two Matrices are Row-Equivalent if one is Obtained from the Other by a Finite Sequence of Row Operations

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Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- Identity Matrix is $n \times n$ Square Matrix whose Row-Vectors and Column Vectors Form an Orthonormal Basis for the n-dimensional Euclidean Vector Space
- Sometimes denoted as $\mathbf{I}_n$ to emphasize its dimension is $n \times n$
- A Permutation Matrix is an Identity Matrix that has Undergone an Arbitrary Series of Row or Column Interchanges

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Trace of a Matrix

- Trace of a Matrix $\mathbf{A}$ is:

$$\mathrm{Tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

- Given two matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$$

$$\mathrm{Tr}(\mathbf{A}+\mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B})$$

$$\mathrm{Tr}(c\mathbf{A}) = c\,\mathrm{Tr}(\mathbf{A})$$

$$\mathrm{Tr}(\mathbf{SAS}^\dagger) = \mathrm{Tr}(\mathbf{S}^\dagger\mathbf{SA}) = \mathrm{Tr}(\mathbf{A})$$
$$\mathrm{Tr}(\mathbf{SBS}^\dagger) = \mathrm{Tr}(\mathbf{S}^\dagger\mathbf{SB}) = \mathrm{Tr}(\mathbf{B})$$

Similarity Transform when A and B are "similar" Matrices

Matrix Determinant

- Determinant of a Matrix is Denoted as:

$$|\mathbf{A}| \qquad\qquad \det(\mathbf{A})$$

- Examples of Determinant Computation:

scalar case

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{A}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

scalar case is absolute value

$$|\mathbf{A}_1| = a_{11} \qquad |\mathbf{A}_2| = a_{11}a_{22} - a_{12}a_{21}$$

$$|\mathbf{A}_3| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrix Operations

- Transpose of a matrix, reflection about the diagonal:

$$\mathbf{A} = \left[a_{ij} \right] \qquad \mathbf{A}^{\mathrm{T}} = \left[a_{ji} \right]$$

- Determinant of $\mathbf{A}$ is Equal to Determinant of $\mathbf{A}^{\mathrm{T}}$
- If Two or More Rows (or columns) of $\mathbf{A}$ are Equivalent then $|\mathbf{A}|=0$
- A Square $n \times n$ Matrix is Triangular When:

$$\forall i > j, a_{ij} = 0 \text{ (upper triangular)}$$
$$\forall i < j, a_{ij} = 0 \text{ (lower triangular)}$$

- Determinant of Triangular Matrix $\mathbf{A}_{\mathrm{tri}}$

$$\det(\mathbf{A}_{\mathrm{tri}}) = |\mathbf{A}_{\mathrm{tri}}| = a_{11} \bullet a_{22} \bullet \ldots \bullet a_{nn}$$

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Rank of a Matrix

- Rank of a Square Matrix, r, is an Integer that is Equal to Number of Linearly Independent Row (Column) Vectors of a Square Matrix
- A Matrix must be Full Rank for a Distinct inverse to Exist
- All Full Rank Matrices may be Converted into Triangular Matrices through Elementary Row Operations
  - allows iterative solution to $\mathbf{Ax}=\mathbf{b}$ through back substitution
- A Full Rank Matrix Must have a non-zero Determinant
- A non-Square Matrix Cannot Have a Rank Larger than $\min(m,n)$

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Linear Independence

- Given:

$$\{c_1, c_2, \cdots, c_m\} \in \mathbb{C} \qquad \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m\} \in \mathbb{R}^n$$

- The set of m Vectors are ***Linearly Independent*** if:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_i\mathbf{a}_i + \cdots + c_m\mathbf{a}_m = \mathbf{0} \Rightarrow c_i = 0 \forall i$$

No Solution for c_i Other Than all Equal 0

- Otherwise, the set of Vectors are Said to be ***Linearly Dependent***
- Linear Independence is a Property of a Specific Subset of Vectors all of dimension n

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Linear Independence Example (TRY THIS)

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- Is the Following set of Vectors Linearly Dependent?:

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

- Check solution for: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

Compute This on Paper

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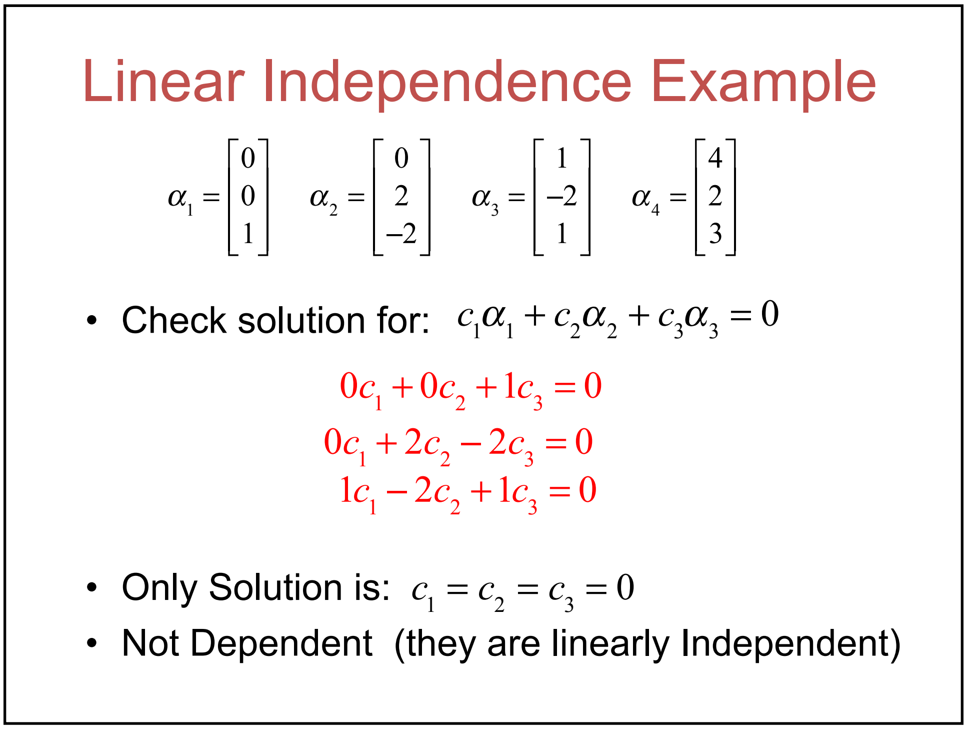

Linear Independence Example

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- Check solution for: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$$0c_1 + 0c_2 + 1c_3 = 0$$
$$0c_1 + 2c_2 - 2c_3 = 0$$
$$1c_1 - 2c_2 + 1c_3 = 0$$

- Only Solution is: $c_1 = c_2 = c_3 = 0$
- Not Dependent (they are linearly Independent)

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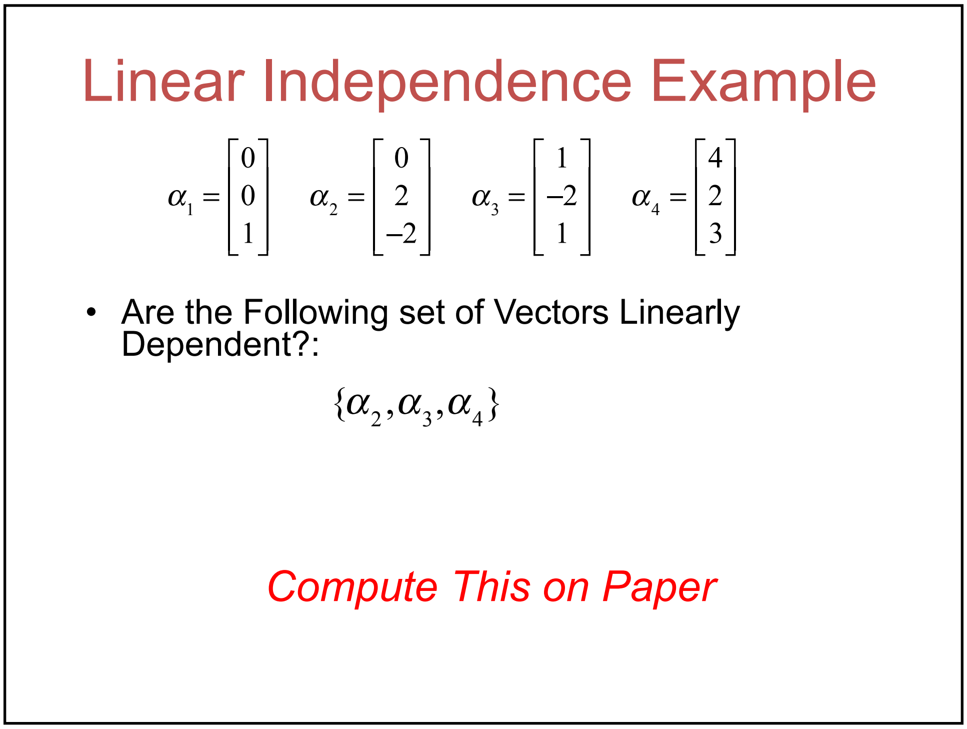

Linear Independence Example

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- Are the Following set of Vectors Linearly Dependent?:

$$\{\alpha_2, \alpha_3, \alpha_4\}$$

Compute This on Paper

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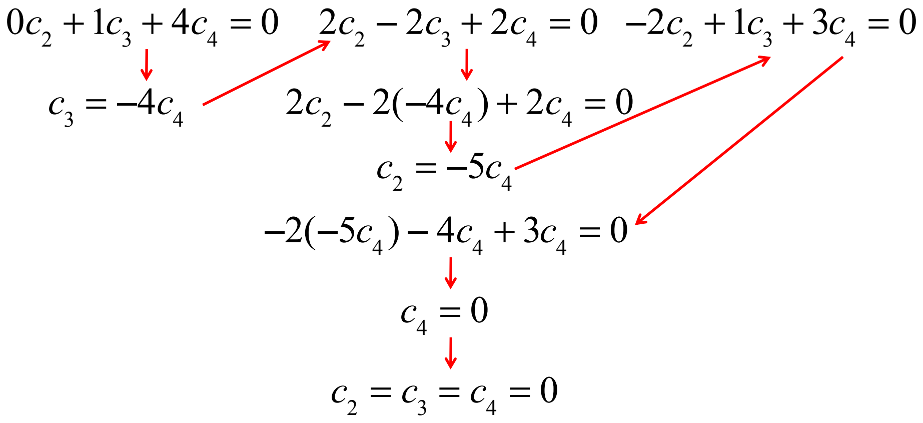
Linear Independence Example



$$0c_2 + 1c_3 + 4c_4 = 0 \qquad 2c_2 - 2c_3 + 2c_4 = 0 \qquad -2c_2 + 1c_3 + 3c_4 = 0$$

$$c_3 = -4c_4$$

$$2c_2 - 2(-4c_4) + 2c_4 = 0$$

$$c_2 = -5c_4$$

$$-2(-5c_4) - 4c_4 + 3c_4 = 0$$

$$c_4 = 0$$

$$c_2 = c_3 = c_4 = 0$$

- Not Dependent (they are linearly Independent)

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- The following sets are Linearly Independent:

$$\{\alpha_1, \alpha_2, \alpha_3\} \qquad \alpha_4 = 9\alpha_1 + 5\alpha_2 + 4\alpha_3$$

$$\{\alpha_2, \alpha_3, \alpha_4\} \quad \alpha_1 = \left(-\frac{5}{9}\right)\alpha_2 + \left(-\frac{4}{9}\right)\alpha_3 + \left(\frac{1}{9}\right)\alpha_4$$

- The following set is Linearly Dependent:

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$