Introduction to Eigensystems

The Eigensystem of a Matrix

- Let $\mathbf{a}_i$ represent a special vector and λ_i represent a special scaling factor with respect to matrix $\mathbf{A}$
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$$\mathbf{A}\mathbf{a}_i = \lambda_i \mathbf{a}_i$$

- In other words, its Maps to a Scaled Version of itself

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- Vector $\mathbf{a}_i$ is an "eigenvector" and scalar λ_i is its associated real/complex non-zero "eigenvalue"

"eigen" is a German word meaning "own," "inherent," "characteristic," or "proper"

Finding Eigenvectors

- By definition, the Eigensystem satisfies:

$$\mathbf{Aa}_i = \lambda_i \mathbf{a}_i$$

$$\mathbf{Aa}_i - \lambda_i \mathbf{a}_i = \mathbf{0}$$

The "0-vector

$$\mathbf{Aa}_i - \lambda_i \mathbf{Ia}_i = \mathbf{0}$$

$$\left(\mathbf{A} - \mathbf{I}\lambda_i\right)\mathbf{a}_i = \mathbf{0}$$

- This system of n equations must be solved for <u>real/complex non-zero values</u> of vectors $\mathbf{a}_i$
- This solution exists if, and only if, the determinant of the coefficient matrix is non-zero
- Thus, we need to solve the "eigen" or characteristic equation for the scalars, λ_i , (components in vector, λ)

$$\left|\mathbf{A} - \mathbf{I}\lambda\right| = 0$$

Characteristic Equation

- Characteristic Equation of a Matrix $\mathbf{A}$ is:

$$c(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = | \mathbf{A} - \lambda\mathbf{I} |$$

- Roots of the Characteristic Equation yield the characteristic values, or eigenvalues, λ_i, of $\mathbf{A}$:

$$| \mathbf{A} - \lambda\mathbf{I} | = 0$$

- Eigenvalues of $\mathbf{A}$ are Scalar Multiples of Eigenvectors, $\{\mathbf{a}_i\}$, of $\mathbf{A}$:
- Eigenvectors of $\mathbf{A}$ are Those Vectors, $\{\mathbf{a}_i\}$, when Mapped by $\mathbf{A}$ are Equivalent to a scaled version of themselves by a Real/Complex non-zero Scale Factor λ_i

Eigensystem Example

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\qquad c(\lambda)=\left|\mathbf{A}-\lambda\mathbf{I}\right|\qquad \mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}5-\lambda&2&2\\2&2-\lambda&1\\2&1&2-\lambda\end{bmatrix}$$

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$$c(\lambda)=\left|\mathbf{A}-\lambda\mathbf{I}\right|=(5-\lambda)\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}-(2)\begin{vmatrix}2&1\\2&2-\lambda\end{vmatrix}+(2)\begin{vmatrix}2&2-\lambda\\2&1\end{vmatrix}$$

$$c(\lambda)=\left|\mathbf{A}-\lambda\mathbf{I}\right|=(5-\lambda)\left[(2-\lambda)^2-1\right]-(2)\left[(4-2\lambda)-2\right]+(2)\left[2-(4-2\lambda)\right]$$

Eigensystem Example

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad c(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| \qquad \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 5-\lambda & 2 & 2 \\ 2 & 2-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

$$c(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = (5-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - (2)\begin{vmatrix} 2 & 1 \\ 2 & 2-\lambda \end{vmatrix} + (2)\begin{vmatrix} 2 & 2-\lambda \\ 2 & 1 \end{vmatrix}$$

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$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - (5-\lambda) - (8-4\lambda) + 4 + 4 - (8-4\lambda)$$

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$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - (5-\lambda) - (8-4\lambda) + 4 + 4 - (8-4\lambda)$$

$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - 5 + \lambda - 8 + 4\lambda + 8 - 8 + 4\lambda$$

Eigensystem Example

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\qquad c(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|\qquad \mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}5-\lambda&2&2\\2&2-\lambda&1\\2&1&2-\lambda\end{bmatrix}$$

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$$c(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|=(5-\lambda)\left[(2-\lambda)^2-1\right]-(2)\left[(4-2\lambda)-2\right]+(2)\left[2-(4-2\lambda)\right]$$

$$c(\lambda)=(5-\lambda)(2-\lambda)^2-(5-\lambda)-(8-4\lambda)+4+4-(8-4\lambda)$$

$$c(\lambda)=(5-\lambda)(2-\lambda)^2-5+\lambda-8+4\lambda+8-8+4\lambda$$

$$c(\lambda)=(5-\lambda)(2-\lambda)^2-13+9\lambda$$

Eigensystem Example

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$$c(\lambda)=(5-\lambda)(4-4\lambda+\lambda^2)-13+9\lambda$$

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$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad c(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| \qquad \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 5-\lambda & 2 & 2 \\ 2 & 2-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

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$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - (5-\lambda) - (8-4\lambda) + 4 + 4 - (8-4\lambda)$$

$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - 5 + \lambda - 8 + 4\lambda + 8 - 8 + 4\lambda$$

$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - 13 + 9\lambda$$

$$c(\lambda) = (5-\lambda)(4 - 4\lambda + \lambda^2) - 13 + 9\lambda$$

$$c(\lambda) = 20 - 20\lambda + 5\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 13 + 9\lambda$$

Eigensystem Example

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$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - (5-\lambda) - (8-4\lambda) + 4 + 4 - (8-4\lambda)$$

$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - 5 + \lambda - 8 + 4\lambda + 8 - 8 + 4\lambda$$

$$c(\lambda) = (5-\lambda)(2-\lambda)^2 - 13 + 9\lambda$$

$$c(\lambda) = (5-\lambda)(4 - 4\lambda + \lambda^2) - 13 + 9\lambda$$

$$c(\lambda) = 20 - 20\lambda + 5\lambda^2 - 4\lambda + 4\lambda^2 - \lambda^3 - 13 + 9\lambda$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

- Next, we find the roots of the characteristic equation to obtain the eigenvalues, so we need to solve:

$$c(\lambda) = 0 \qquad -\lambda^3 + 9\lambda^2 - 15\lambda + 7 = 0$$

- A "good" guess is to divide the λ^0 coefficient by leading (λ^3) coefficient: 7/(-1)=-7, try (λ-7):

$$\frac{-\lambda^3 + 9\lambda^2 - 15\lambda + 7}{\lambda - 7} = -\lambda^2 + 2\lambda - 1$$

- It worked! Zero remainder, so first factor is: (λ-7)
- Next factor the quotient (note that leading -1 is irrelevant):

$$-\lambda^2 + 2\lambda - 1 = -\left(\lambda^2 - 2\lambda + 1\right) = -(\lambda - 1)(\lambda - 1)$$

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

- We have found the binomial factors of the characteristic equation: $c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7 = -(\lambda - 7)(\lambda - 1)(\lambda - 1)$
- It never hurts to double check by multiplying it back out:

$$c(\lambda) = -(\lambda - 7)(\lambda - 1)(\lambda - 1) = -(\lambda - 7)\left(\lambda^2 - 2\lambda + 1\right)$$

$$= -\left(\lambda^3 - 2\lambda^2 + \lambda - 7\lambda^2 + 14\lambda - 7\right) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

- Thus, we have the eigenvalues (spectrum) of the matrix $\mathbf{A}$:

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7, 1, 1\}$$

- Note that we have the eigenvalue 7 with multiplicity 1 and the eigenvalue 1 with multiplicity 2

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7, 1, 1\}$$

$$\lambda_1 = 7$$

- Next, find the eigenvectors, start with λ_1=7:

$$\mathbf{A}\mathbf{a}_1 = 7\mathbf{a}_1$$

$$\mathbf{A}\mathbf{a}_1 - 7\mathbf{a}_1 = \mathbf{0}$$

$$(\mathbf{A} - 7\mathbf{I})\mathbf{a}_1 = \mathbf{0}$$

$$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2a_x + 2a_y + 2a_z = 0$$

$$2a_x - 5a_y + a_z = 0$$

$$2a_x + a_y - 5a_z = 0$$

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7, 1, 1\}$$

$$\lambda_1 = 7$$

$$-2a_x + 2a_y + 2a_z = 0$$

$$2a_x - 5a_y + a_z = 0$$

$$2a_x + a_y - 5a_z = 0$$

- We want the non-trivial solution. One way is to use Gaussian elimination.
- First, write the augmented matrix representing the equations for λ_1=7:

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 2 & -5 & 1 & 0 \\ 2 & 1 & -5 & 0 \end{array}\right]$$

- Next, perform elementary row operations attempt to transform the leftmost side to the identity matrix.

Eigensystem Example (cont.)

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\qquad \begin{gathered}c(\lambda)=-\lambda^3+9\lambda^2-15\lambda+7\\ \{\lambda_1,\lambda_2,\lambda_3\}=\{7,1,1\}\\ \lambda_1=7\end{gathered}\qquad \left[\begin{array}{ccc|c}-2&2&2&0\\2&-5&1&0\\2&1&-5&0\end{array}\right]$$

- Perform elementary row operations:

$$\left[\begin{array}{ccc|c}-2&2&2&0\\2&-5&1&0\\2&1&-5&0\end{array}\right]\xrightarrow{\text{R1+R2}\to\text{R2}}\left[\begin{array}{ccc|c}-2&2&2&0\\0&-3&3&0\\2&1&-5&0\end{array}\right]\xrightarrow{\text{R1+R3}\to\text{R3}}\left[\begin{array}{ccc|c}-2&2&2&0\\0&-3&3&0\\0&3&-3&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}-2&2&2&0\\0&-3&3&0\\0&3&-3&0\end{array}\right]\xrightarrow{-(1/2)\text{R1}\to\text{R1}}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&-3&3&0\\0&3&-3&0\end{array}\right]\xrightarrow{\text{R2+R3}\to\text{R3}}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&-3&3&0\\0&0&0&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&-1&-1&0\\0&-3&3&0\\0&0&0&0\end{array}\right]\xrightarrow{-(1/3)\text{R2}\to\text{R2}}\left[\begin{array}{ccc|c}1&-1&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]\xrightarrow{\text{R1+R2}\to\text{R1}}\left[\begin{array}{ccc|c}1&0&-2&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

Eigensystem Example (cont.)

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\qquad \begin{gathered}c(\lambda)=-\lambda^3+9\lambda^2-15\lambda+7\\ \{\lambda_1,\lambda_2,\lambda_3\}=\{7,1,1\}\\ \lambda_1=7\end{gathered}\qquad \left[\begin{array}{ccc|c}1&0&-2&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$$

- These equations define the family of eigenvectors for λ_1=7:

$$a_x-2a_z\Rightarrow a_x=2a_z$$
$$a_y-a_z\Rightarrow a_y=a_z$$

- Solve the equations with a parameter, a_z=s, for the reduced all-zero row (third a_z row of reduced matrix)
- Parameterized eigenvector for λ_1=7:

$$\mathbf{a}_1=\begin{bmatrix}2s\\s\\s\end{bmatrix}\qquad \mathbf{Aa}_1=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\begin{bmatrix}2s\\s\\s\end{bmatrix}=\begin{bmatrix}14s\\7s\\7s\end{bmatrix}=(7)\begin{bmatrix}2s\\s\\s\end{bmatrix}$$

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7,1,1\}$$

$$\lambda_2 = \lambda_3 = 1$$

- Next, find the eigenvectors, start with $\lambda_2=\lambda_3=1$:

$$\mathbf{A}\mathbf{a}_2 = \mathbf{a}_2 \qquad \mathbf{A}\mathbf{a}_3 = \mathbf{a}_3$$

$$\mathbf{A}\mathbf{a}_2 - \mathbf{a}_2 = \mathbf{0} \qquad \mathbf{A}\mathbf{a}_3 - \mathbf{a}_3 = \mathbf{0}$$

$$(\mathbf{A}-\mathbf{I})\mathbf{a}_2 = \mathbf{0} \qquad (\mathbf{A}-\mathbf{I})\mathbf{a}_3 = \mathbf{0}$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4a_x + 2a_y + 2a_z = 0$$
$$2a_x + a_y + a_z = 0$$
$$2a_x + a_y + a_z = 0$$

Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7,1,1\}$$

$$\lambda_2 = \lambda_3 = 1$$

- Use Gaussian elimination to find the non-trivial solution for $\lambda_2=\lambda_3=1$:
- Write the augmented matrix representing the equations for $\lambda_2=\lambda_3=1$:

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

- Perform elementary row operations attempt to transform the leftmost side to the identity matrix.

Eigensystem Example (cont.)

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}$$

$$c(\lambda)=-\lambda^3+9\lambda^2-15\lambda+7$$

$$\{\lambda_1,\lambda_2,\lambda_3\}=\{7,1,1\}$$

$$\lambda_2=\lambda_3=1$$

$$\left[\begin{array}{ccc|c}4&2&2&0\\2&1&1&0\\2&1&1&0\end{array}\right]$$

- Perform elementary row operations:

$$\left[\begin{array}{ccc|c}4&2&2&0\\2&1&1&0\\2&1&1&0\end{array}\right]\xrightarrow{\text{R2-(1/2)R1}\Rightarrow\text{R2}}\left[\begin{array}{ccc|c}4&2&2&0\\0&0&0&0\\2&1&1&0\end{array}\right]\xrightarrow{\text{R3-(1/2)R1}\Rightarrow\text{R3}}\left[\begin{array}{ccc|c}4&2&2&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}4&2&2&0\\0&0&0&0\\0&0&0&0\end{array}\right]\xrightarrow{\text{(1/4)R1}\Rightarrow\text{R1}}\left[\begin{array}{ccc|c}1&\frac{1}{2}&\frac{1}{2}&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

- This equation defines the family of eigenvectors for $\lambda_2=\lambda_3=1$:

$$a_x+\frac{1}{2}a_y+\frac{1}{2}a_z\Rightarrow a_x=-\frac{1}{2}a_y-\frac{1}{2}a_z$$

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Eigensystem Example (cont.)

$$\mathbf{A}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}$$

$$c(\lambda)=-\lambda^3+9\lambda^2-15\lambda+7$$

$$\{\lambda_1,\lambda_2,\lambda_3\}=\{7,1,1\}$$

$$\lambda_2=\lambda_3=1$$

$$\left[\begin{array}{ccc|c}4&2&2&0\\2&1&1&0\\2&1&1&0\end{array}\right]$$

- Solve the equation with parameters, $a_y=s$, $a_z=t$, for the reduced all-zero rows (second/third a_y/a_z rows of reduced matrix)
- Parameterized eigenvector for $\lambda_2=\lambda_3=1$:

$$\mathbf{a}_{2,3}=\begin{bmatrix}-\frac{1}{2}s-\frac{1}{2}t\\s\\t\end{bmatrix}\qquad\mathbf{Aa}_{2,3}=\begin{bmatrix}5&2&2\\2&2&1\\2&1&2\end{bmatrix}\begin{bmatrix}-\frac{1}{2}s-\frac{1}{2}t\\s\\t\end{bmatrix}=(1)\begin{bmatrix}-\frac{1}{2}s-\frac{1}{2}t\\s\\t\end{bmatrix}$$

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Eigensystem Example (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$
$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7,1,1\}$$
$$\lambda_2 = \lambda_3 = 1$$

$$\mathbf{a}_{2,3} = \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix}$$

- Parameterized eigenvector for $\lambda_2 = \lambda_3 = 1$:

$$\mathbf{a}_{2,3} = \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix}$$

$$\mathbf{Aa}_{2,3} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix} = (1) \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix}$$

Eigensystem Example Summary

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$c(\lambda) = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$
$$\{\lambda_1, \lambda_2, \lambda_3\} = \{7,1,1\}$$
$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$$

- Parameterized eigenvectors:

$$\mathbf{a}_1 = \begin{bmatrix} 2s \\ s \\ s \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix} \qquad \mathbf{a}_3 = \begin{bmatrix} -\tfrac{1}{2}s - \tfrac{1}{2}t \\ s \\ t \end{bmatrix}$$

- Rank, r, of matrix $\mathbf{A}$ is number of unique eigenvalues:

$$r = 2$$

- Determinate, $|\mathbf{A}|$, is product of eigenvalues:

$$|\mathbf{A}| = \det(\mathbf{A}) = (\lambda_1)(\lambda_2)(\lambda_3) = (7)(1)(1) = 7$$

- Trace, $\mathrm{tr}(\mathbf{A})$, is the sum of eigenvalues:

$$\mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 = 7 + 1 + 1 = 9$$

Eigenvectors/Eigenvalues Facts

- Only Exist for Square Matrices
- If the same eigenvalue, λ_i, is repeated m_i times, then that eigenvalue is said to have an "algebraic multiplicity" of m_i
- The number of Unique eigenvalues, r, of a Matrix $\mathbf{A}$ is Equivalent to the Rank of $\mathbf{A}$
- Determinate of $\mathbf{A}$: $\det(\mathbf{A}) = |\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$
- Sum of eigenvalues is Trace of $\mathbf{A}$: $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \boldsymbol{\lambda}_i$
- Invertible $\mathbf{A}$: $r=n$ and eigenvalues of $\mathbf{A}^{-1}$ are $\{\lambda_i^{-1}\}$
- The set of eigenvalues of $\mathbf{A}$ are the "*spectrum*" of $\mathbf{A}$

Eigenvalues are the "vital signs" of a Matrix !!!!

(optional)

Eigensystems

https://www.youtube.com/watch?v=ue3yoeZvt8E (4:00)

Introduction to Operators

Mathematical Operator

- Generally a mapping that acts on elements of a space to produce elements of another space
  - common operator is a linear map that acts on a vector space (others are possible)
  - often means actions on vector spaces of functions
  - general linear operator takes the form of a matrix $\mathbf{A}$ and obeys the following where $\mathbb{W}$ and $\mathbb{V}$ are vector spaces and $\mathbf{A}$: $\mathbb{W} \rightarrow \mathbb{V}$ and where (α, β are scalars and $\mathbf{x}$, $\mathbf{y}$ are elements of $\mathbb{W}$):

$$\mathbf{A}\left(\alpha\mathbf{x}+\beta\mathbf{y}\right)=\alpha\mathbf{A}\mathbf{x}+\beta\mathbf{A}\mathbf{y}$$

Linear Operator/Map

- Matrix that Maps one Element of a Vector Space to Another and Preserves Addition and Scalar Multiplication

$$\hat{\mathbf{A}}(\mathbf{x}+\mathbf{y}) = \hat{\mathbf{A}}\mathbf{x}+\hat{\mathbf{A}}\mathbf{y} \qquad \hat{\mathbf{A}}(\alpha\mathbf{x}) = \alpha\hat{\mathbf{A}}\mathbf{x}$$

- To Emphasize a Matrix is an Operator, it often "wears a hat" (*eg*. $\hat{\mathbf{A}}$)
  - not necessary, just used for emphasis
- Many Different Linear Operators used in Quantum Mechanics
- The "hat" Notation is not always used

Mathematical Operators (optional)

General Notions:
https://www.youtube.com/watch?v=LtFsf-TR_M0 (5:49)

Matrices as Operators on Vectors:
https://www.youtube.com/watch?v=f74DQnYjJes (14:14)

Adjoint Operator

- The "adjoint" of a Matrix $\mathbf{A}$ is the "conjugate-transpose" or "transpose-conjugate" of $\mathbf{A}$ Denoted by $\mathbf{A}^{\dagger}$using the Superscript "dagger" Symbol $\dagger$
- Applicable to Vectors and Matrices
  - Vector: $\mathbf{a}^{\dagger} = \left(\mathbf{a}^{*}\right)^{\mathrm{T}} = \left(\mathbf{a}^{\mathrm{T}}\right)^{*}$
  - Matrix: $\mathbf{A}^{\dagger} = \left(\mathbf{A}^{*}\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{*}$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}$$

$$\text{Find}: \quad \mathbf{A}^{\dagger} = ?$$

Compute This on Paper

Adjoint Operator Example

- The "conjugate-transpose" or "transpose-conjugate" Denoted by Superscript "dagger" Symbol $\dagger$
- Applicable to Vectors and Matrices
  - Vector: $\mathbf{a}^{\dagger} = \left(\mathbf{a}^{*}\right)^{\mathrm{T}} = \left(\mathbf{a}^{\mathrm{T}}\right)^{*}$
  - Matrix: $\mathbf{A}^{\dagger} = \left(\mathbf{A}^{*}\right)^{\mathrm{T}} = \left(\mathbf{A}^{\mathrm{T}}\right)^{*}$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}$$

$$\mathbf{A}^{\dagger} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}^{\dagger} = \left(\begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}^{*}\right)^{\mathrm{T}}$$

$$\mathbf{A}^{\dagger} = \begin{bmatrix} 1+5i & 1-i \\ 1-3i & -7i \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1+5i & 1-3i \\ 1-i & -7i \end{bmatrix}$$

Adjoint Operator Properties

- Definition and Properties of the Adjoint

$\mathbf{A}^{\dagger}_{ij} = \mathbf{A}^{*}_{ji} \;\; \forall \; \{(i,j)|i = 0,1,\cdots,n-1 \text{ and } j = 0,1,\cdots,n-1\}$

$(\mathbf{AB})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}$

$\left(\mathbf{A}^{\dagger}\right)^{\dagger} = \mathbf{A}$

$(\mathbf{A}+\mathbf{B})^{\dagger} = \mathbf{A}^{\dagger} + \mathbf{B}^{\dagger}$

$(c\mathbf{A})^{\dagger} = c^{*}\mathbf{A}^{\dagger}$

- Theorem of the Adjoint: For every pair of vector, $\mathbf{x}$ and $\mathbf{y}$, and every operator $\mathbf{A}$, the inner product relations hold:

$$\mathbf{y}\cdot\mathbf{Ax} = \mathbf{A}^{\dagger}\mathbf{y}\cdot\mathbf{x} \qquad \mathbf{Ay}\cdot\mathbf{x} = \mathbf{y}\cdot\mathbf{A}^{\dagger}\mathbf{x}$$

- In Dirac's notation (explained later) these relations are:

$$\langle y|\mathbf{A}x\rangle = \langle \mathbf{A}^{\dagger}y|x\rangle \qquad \langle \mathbf{A}y|x\rangle = \langle y|\mathbf{A}^{\dagger}x\rangle$$

Special Forms of Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{A} = \left[a_{ij}\right]_{n\times n}$$

- Symmetric if: $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$
- Hermitian if: $\mathbf{A}^{\dagger} = \mathbf{A}$
- Normal if: $\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}\mathbf{A}^{\dagger}$
- Orthogonal if: $\mathbf{A}^{\mathrm{T}}\,\mathbf{A} = \mathbf{I}$
- Unitary if: $\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{I}$
- Lower Triangular if: $a_{ij} = 0 \;\; \forall i > j$
- Upper Triangular if: $a_{ij} = 0 \;\; \forall i < j$
- Diagonal if: $a_{ij} = 0 \;\; \forall i \neq j$

Normal Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{A} = \left[a_{ij} \right]_{n\times n}$$

- $\mathbf{A}$ is a normal matrix, by definition, if it commutes under direct multiplication: $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}$
- The definition implies that $\mathbf{A}$ must be square, $n\times n$
- Since $\mathbf{A}$ is square, it maps/transforms/operates on vectors within the same vector space
- If $\mathbf{A}$ and $\mathbf{B}$ are normal, then so are the product matrix $\mathbf{AB}$ and the summation matrix, $\mathbf{A}+\mathbf{B}$
  - $\mathbf{A}$ and $\mathbf{B}$ are *simultaneously diagonalizable*, $\mathbf{SAS}^{-1}=\mathbf{SBS}^{-1}=\mathbf{D}$, for some invertible *similarity matrix* $\mathbf{S}$ and diagonal matrix, $\mathbf{D}$

Hermitian Matrices

- Consider a matrix, $\hat{\mathbf{H}}$, that is Hermitian
  - Hermitian means that matrix is a complex square matrix that is equal to its own adjoint
  - conjugate transpose of $\hat{\mathbf{H}}=[h_{ij}]_{n\times n}$: $\left(\hat{\mathbf{H}}^{\mathrm{T}}\right)^{*} = \left(\hat{\mathbf{H}}^{*}\right)^{\mathrm{T}} = \hat{\mathbf{H}}^{\dagger}$
  - Hermitian means: $\hat{\mathbf{H}} = \hat{\mathbf{H}}^{\dagger}$ $\quad \left[h_{ij}\right] = \left[h_{ji}^{*}\right]$
  - Hermitian matrices are not always invertible
  - Real Hermitian matrices are symmetric
- Spectral Decomposition (defined later) has Desirable Properties for Hermitian matrices
- Spectral Decomposition is useful in Describing the Hamiltonian Operator actions for Quantum Mechanical Systems

Hermitian Matrices are Normal

- Consider a matrix, $\hat{\mathbf{H}}$, that is Hermitian

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}^{\dagger}$$

$$\hat{\mathbf{H}}\hat{\mathbf{H}} = \hat{\mathbf{H}}\hat{\mathbf{H}}^{\dagger} = \hat{\mathbf{H}}^{\dagger}\hat{\mathbf{H}} = \hat{\mathbf{H}}^{2}$$

- Real-valued Symmetric Matrices are Hermitian and also Normal

Eigensystem Facts for Hermitian $\mathbf{H}$

- $\mathbf{H}$ is full rank/invertible only when $r=n$
  - *eg*., All zeros matrix, $\mathbf{0}$, is Hermitian
- All eigenvalues of $\mathbf{H}$ are real $\lambda_i \in \mathbb{R}$
- The eigenvectors $\{\mathbf{h}_i | i=1,n\}$ are identical for $\mathbf{H}$ and $\mathbf{H}^{\dagger}$, since $\mathbf{H}=\mathbf{H}^{\dagger}$
- The nontrivial eigenvectors $\{\mathbf{h}_i | i=1,n\}$ are orthogonal for $\mathbf{H}$: $\mathbf{h}_i\mathbf{h}_j^{\dagger} = \mathbf{h}_i \bullet \mathbf{h}_j = 0, \forall i \neq j$
- Inner (dot is used here) product property of Hermitian matrix:

$$\mathbf{a} \bullet \mathbf{Hb} = \mathbf{Ha} \bullet \mathbf{b} \qquad \mathbf{a}\left(\mathbf{Hb}\right)^{\dagger} = \left(\mathbf{Ha}\right)^{\dagger}\mathbf{b}$$

Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Eigenvector Definition:

$$\mathbf{Hx} = \lambda\mathbf{x}$$

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- Multiply both sides by adjoint of eigenvector:

$$\mathbf{x}^{\dagger}\mathbf{Hx} = \mathbf{x}^{\dagger}\lambda\mathbf{x}$$

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- Recognizing the dot product yields norm:

$$\mathbf{x}^\dagger\mathbf{Hx} = \lambda\|\mathbf{x}\|$$

Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Recognizing the dot product yields norm:

$$\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x} = \lambda\|\mathbf{x}\|$$

- Take adjoint of both sides of this equation:

$$\left(\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x}\right)^{\dagger} = (\lambda\|\mathbf{x}\|)^{\dagger}$$

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$$(\mathbf{x})^{\dagger}\mathbf{H}^{\dagger}\left(\mathbf{x}^{\dagger}\right)^{\dagger} = \lambda^{*}\|\mathbf{x}\|^{*}$$

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- Norm is real since $\mathbf{x}^{\dagger}\mathbf{x} \in \mathbb{R}$:

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- Since $\mathbf{H}$ is Hermitian, $\mathbf{H}^{\dagger} = \mathbf{H}$**:**

$$(\mathbf{x})^{\dagger}\mathbf{H}\left(\mathbf{x}^{\dagger}\right)^{\dagger} = \lambda^{*}\|\mathbf{x}\|$$

Eigenvalues of Hermitian $\mathbf{H}$ are Real

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$$\mathbf{x}^\dagger\mathbf{H}\mathbf{x} = \lambda^*\|\mathbf{x}\| \qquad (1)$$

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$$\mathbf{x}^\dagger\mathbf{H}\mathbf{x} = \lambda^*\|\mathbf{x}\| \qquad (1)$$

- Recognize Definition of Eigenvector in LHS:

$$\mathbf{x}^\dagger(\mathbf{H}\mathbf{x}) = \mathbf{x}^\dagger(\lambda\mathbf{x}) = \lambda\mathbf{x}^\dagger\mathbf{x} = \lambda\|\mathbf{x}\| \qquad (2)$$

Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Norm is real since $\mathbf{x}^{\dagger}\mathbf{x} \in \mathbb{R}$:

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- Since $\mathbf{H}$ is Hermitian, $\mathbf{H}^{\dagger} = \mathbf{H}$:

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- Equating Equations (1) and (2):

$$\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x} = \lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$$

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Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Equating Equations (1) and (2):

$$\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x} = \lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$$

- For non-trivial case, $\|\mathbf{x}\| > 0$. Divide both sides of $\lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$ by $\|\mathbf{x}\|$:

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Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Equating Equations (1) and (2):

$$\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x} = \lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$$

- For non-trivial case, $\|\mathbf{x}\| > 0$. Divide both sides of $\lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$ by $\|\mathbf{x}\|$:

$$\lambda^{*} = \lambda$$

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Eigenvalues of Hermitian $\mathbf{H}$ are Real

- Equating Equations (1) and (2):

$$\mathbf{x}^{\dagger}\mathbf{H}\mathbf{x} = \lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$$

- For non-trivial case, $\|\mathbf{x}\| > 0$. Divide both sides of $\lambda^{*}\|\mathbf{x}\| = \lambda\|\mathbf{x}\|$ by $\|\mathbf{x}\|$:

$$\lambda^{*} = \lambda$$

- This can only hold if the eigenvalues are real.

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Unitary Matrices

- Consider matrix, $\mathbf{U}$, that is Unitary
  - Unitary means that the conjugate transpose of $\mathbf{U}$ is its own inverse ($\mathbf{U}$ is may or may not be Hermitian)

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$$

  - Thus, unitaries are normal matrices

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$$

  - Property giving the name "unitary:"

$$\det(\mathbf{U}) = |\mathbf{U}| = 1$$

  - Colum/Row vectors are Orthonormal
  - Matrix exponential of a Hermitian matrix, $\hat{\mathbf{H}}$ is unitary:

$$\mathbf{U} = e^{i\hat{\mathbf{H}}}$$

$$\mathbf{U}^{\dagger} = e^{-i\hat{\mathbf{H}}^{\dagger}} = e^{-i\hat{\mathbf{H}}}$$

$$\mathbf{U}\mathbf{U}^{\dagger} = \left(e^{i\hat{\mathbf{H}}}\right)\left(e^{-i\hat{\mathbf{H}}}\right) = \mathbf{I}$$

Unitary Matrices

- Consider matrix, $\mathbf{U}$, that is Unitary

$$\mathbf{U}^{-1} = \mathbf{U}^{\dagger} \quad \mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I} \quad \det(\mathbf{U}) = |\mathbf{U}| = 1$$

  - Colum/Row vectors are Orthonormal
- A unitary, $\mathbf{U}$**,** *can* be formed as the matrix exponential of a Hermitian matrix, $\hat{\mathbf{H}}$:

$$\mathbf{U} = e^{i\hat{\mathbf{H}}}$$

- $\mathbf{U}$ may also be Hermitian when the matrix exponent of a Hermitian matrix is also a Hermitian matrix
  - Eigenvalues lie on unit circle, eigenvalue magnitudes are 1; n roots of Unity in complex plane

Matrix Exponentiation Identities

- In general, a power series for an $n \times n$ real/complex matrix:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k$$

- Special case when $\mathbf{A}$ is Hermitian/Unitary <u>and</u> $\mathbf{AA}=\mathbf{A}^2=\mathbf{I}$ and β is a real number:

$$e^{\beta \mathbf{A}} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!} \mathbf{A}^k \qquad e^{\beta \mathbf{A}} = \cosh(\beta)\mathbf{I} + \sinh(\beta)\mathbf{A}$$

$$e^{i\beta \mathbf{A}} = \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!} \mathbf{A}^k \qquad e^{i\beta \mathbf{A}} = \cos(\beta)\mathbf{I} + i\sin(\beta)\mathbf{A}$$

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Matrix Exponentiation: with Real Scalar

- In general, a power series for an $n \times n$ real/complex matrix: $e^{\beta \mathbf{A}} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!} \mathbf{A}^k$
- Special case when $\mathbf{A}$ is Hermitian and $\mathbf{AA}=\mathbf{A}^2=\mathbf{I}$ and β is a real number
- Expanding according to power series:

$$e^{\beta \mathbf{A}} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!} \mathbf{A}^k = \mathbf{I} + \beta\mathbf{A} + \frac{\beta^2}{2!}\mathbf{I} + \frac{\beta^3}{3!}\mathbf{A} + \cdots$$

$$= \left(1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots\right)\mathbf{I} + \left(\beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right)\mathbf{A}$$

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Matrix Exponentiation: with Real Scalar (cont.)

$$e^{\beta\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!}\mathbf{A}^k = \mathbf{I} + \beta\mathbf{A} + \frac{\beta^2}{2!}\mathbf{I} + \frac{\beta^3}{3!}\mathbf{A} + \cdots$$

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Power Series for an Exponential of Real Scalar, β:

$$e^{\beta} = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} = 1+\frac{\beta}{1!}+\frac{\beta^2}{2!}+\frac{\beta^3}{3!}+\cdots+\frac{\beta^n}{n!}+\cdots$$

$$e^{-\beta} = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} = 1-\frac{\beta}{1!}+\frac{\beta^2}{2!}-\frac{\beta^3}{3!}+\cdots+(-1)^n\frac{\beta^n}{n!}+\cdots$$

Definition hyperbolic cosine of Real Scalar, β:

$$\cosh(\beta) = \frac{e^{\beta}+e^{-\beta}}{2} = \frac{1}{2}\left(1+\frac{\beta}{1!}+\frac{\beta^2}{2!}+\frac{\beta^3}{3!}+\cdots+\frac{\beta^n}{n!}+\cdots\right)+\frac{1}{2}\left(1-\frac{\beta}{1!}+\frac{\beta^2}{2!}-\frac{\beta^3}{3!}+\cdots+(-1)^n\frac{\beta^n}{n!}+\cdots\right)$$

$$\frac{e^{\beta}+e^{-\beta}}{2} = \left(1+\frac{\beta^2}{2!}+\frac{\beta^4}{4!}+\cdots\right)$$

Matrix Exponentiation: with Real Scalar (cont.)

$$e^{\beta\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(\beta)^k}{k!}\mathbf{A}^k = \mathbf{I} + \beta\mathbf{A} + \frac{\beta^2}{2!}\mathbf{I} + \frac{\beta^3}{3!}\mathbf{A} + \cdots$$

$$= \left(1+\frac{\beta^2}{2!}+\frac{\beta^4}{4!}+\cdots\right)\mathbf{I} + \left(\beta+\frac{\beta^3}{3!}+\frac{\beta^5}{5!}+\cdots\right)\mathbf{A}$$

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$$\frac{e^{\beta}-e^{-\beta}}{2} = \left(\frac{\beta}{1!}+\frac{\beta^3}{3!}+\frac{\beta^5}{5!}+\cdots\right)$$

Matrix Exponentiation: with Real Scalar (cont.)

$$e^{\beta\mathbf{A}} = \sum_{k=0}^{\infty}\frac{(\beta)^k}{k!}\mathbf{A}^k = \mathbf{I} + \beta\mathbf{A} + \frac{\beta^2}{2!}\mathbf{I} + \frac{\beta^3}{3!}\mathbf{A} + \cdots$$

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$$\frac{e^{\beta}+e^{-\beta}}{2} = \left(1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots\right)$$

Definition hyperbolic sine of Real Scalar, β:

$$\sinh(\beta) = \frac{e^{\beta}-e^{-\beta}}{2} = \left(1 + \frac{\beta}{1!} + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots + \frac{\beta^n}{n!} + \cdots\right) - \left(1 - \frac{\beta}{1!} + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \cdots + (-1)^n\frac{\beta^n}{n!} + \cdots\right)$$

$$\frac{e^{\beta}-e^{-\beta}}{2} = \left(\frac{\beta}{1!} + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right)$$

Matrix Exponentiation: with Real Scalar (cont.)

$$e^{\beta\mathbf{A}} = \sum_{k=0}^{\infty}\frac{(\beta)^k}{k!}\mathbf{A}^k = \mathbf{I} + \beta\mathbf{A} + \frac{\beta^2}{2!}\mathbf{I} + \frac{\beta^3}{3!}\mathbf{A} + \cdots$$

$$= \left(1 + \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots\right)\mathbf{I} + \left(\beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \cdots\right)\mathbf{A}$$

Substituting hyperbolic cosine, sine of β into above:

$\therefore$ $$e^{\beta\mathbf{A}} = \cosh(\beta)\mathbf{I} + \sinh(\beta)\mathbf{A}$$ (QED)

(review)

Euler's Identity and Series Definitions

https://www.youtube.com/watch?v=sKtloBAuP74 (14:30)

Matrix Exponentiation: complex scalar

$$e^{i\beta\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!}\mathbf{A}^k$$

- Previous result:

$$e^{\beta\mathbf{A}} = \cosh(\beta)\mathbf{I} + \sinh(\beta)\mathbf{A}$$

- Changing argument from real, β, to complex, $i\beta$

$$\sinh(i\beta) = \frac{1}{2}\left(e^{i\beta} - e^{-i\beta}\right) \qquad \cosh(i\beta) = \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right)$$

$$e^{i\beta} = \cos(\beta) + i\sin(\beta) \qquad e^{-i\beta} = \cos(\beta) - i\sin(\beta)$$

$$\sinh(i\beta) = \frac{1}{2}\left[\cos(\beta) + i\sin(\beta) - \cos(\beta) + i\sin(\beta)\right] = i\sin(\beta)$$

$$\cosh(i\beta) = \frac{1}{2}\left[\cos(\beta) + i\sin(\beta) + \cos(\beta) - i\sin(\beta)\right] = \cos(\beta)$$

$$\therefore \quad e^{i\beta\mathbf{A}} = \cosh(i\beta)\mathbf{I} + \sinh(i\beta)\mathbf{A} = \cos(\beta)\mathbf{I} + i\sin(\beta)\mathbf{A} \quad \text{(QED)}$$

Eigenvalues of a Unitary Matrix have unity magnitude

- Let $\mathbf{a}_i$ represent an eigenvector and λ_i represent the corresponding eigenvalue of $\mathbf{U}$

$$\mathbf{U}\mathbf{a}_i = \lambda_i \mathbf{a}_i$$

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- Let $\mathbf{a}_i$ represent an eigenvector and λ_i represent the corresponding eigenvalue of $\mathbf{U}$

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- Taking the adjoint of both sides:

$$\left(\mathbf{U}\mathbf{a}_i\right)^\dagger = \left(\lambda_i \mathbf{a}_i\right)^\dagger$$

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$$\mathbf{U}\mathbf{a}_i = \lambda_i \mathbf{a}_i$$

- Taking the adjoint of both sides:

$$(\mathbf{U}\mathbf{a}_i)^\dagger = (\lambda_i \mathbf{a}_i)^\dagger$$
$$\mathbf{a}_i^\dagger \mathbf{U}^\dagger = \lambda_i^* \mathbf{a}_i^\dagger$$

Eigenvalues of a Unitary Matrix have unity magnitude

- Let $\mathbf{a}_i$ represent an eigenvector and λ_i represent the corresponding eigenvalue of $\mathbf{U}$

$$\mathbf{U}\mathbf{a}_i = \lambda_i \mathbf{a}_i$$

- Taking the adjoint of both sides:

$$(\mathbf{U}\mathbf{a}_i)^\dagger = (\lambda_i \mathbf{a}_i)^\dagger$$
$$\mathbf{a}_i^\dagger \mathbf{U}^\dagger = \lambda_i^* \mathbf{a}_i^\dagger$$

- Multiply each side of previous equation with each side of top equation:

$$\left(\mathbf{a}_i^\dagger \mathbf{U}^\dagger\right)\left(\mathbf{U}\mathbf{a}_i\right) = \left(\lambda_i^* \mathbf{a}_i^\dagger\right)\left(\lambda_i \mathbf{a}_i\right)$$

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$$\mathbf{a}_i^\dagger \left(\mathbf{U}^\dagger \mathbf{U}\right)\mathbf{a}_i = \lambda_i^* \lambda_i \mathbf{a}_i^\dagger \mathbf{a}_i$$

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$$\mathbf{a}_i^\dagger \left(\mathbf{U}^\dagger \mathbf{U}\right)\mathbf{a}_i = \lambda_i^* \lambda_i \mathbf{a}_i^\dagger \mathbf{a}_i$$
$$\mathbf{a}_i^\dagger \left(\mathbf{I}\right)\mathbf{a}_i = \lambda_i^* \lambda_i \mathbf{a}_i^\dagger \mathbf{a}_i$$

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$$\mathbf{a}_i^\dagger \mathbf{a}_i = \left(\lambda_i^* \lambda_i\right)\mathbf{a}_i^\dagger \mathbf{a}_i$$

Eigenvalues of a Unitary Matrix have unity magnitude (cont.)

- Because:

$$\mathbf{a}_i^\dagger \mathbf{a}_i = \left(\lambda_i^* \lambda_i\right)\mathbf{a}_i^\dagger \mathbf{a}_i$$
$$\left(\lambda_i^* \lambda_i\right)\mathbf{a}_i^\dagger \mathbf{a}_i - \mathbf{a}_i^\dagger \mathbf{a}_i = 0$$

Eigenvalues of a Unitary Matrix have unity magnitude (cont.)

- Because:

$$\mathbf{a}_i^\dagger \mathbf{a}_i = \left(\lambda_i^* \lambda_i\right) \mathbf{a}_i^\dagger \mathbf{a}_i$$

$$\left(\lambda_i^* \lambda_i\right) \mathbf{a}_i^\dagger \mathbf{a}_i - \mathbf{a}_i^\dagger \mathbf{a}_i = 0$$

$$\left(\lambda_i^* \lambda_i - 1\right) \mathbf{a}_i^\dagger \mathbf{a}_i = 0$$

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- We know that the inner product of an eigenvector with itself (square of its norm) cannot be 0, since $\mathbf{U}$ is full rank, thus

$$\left(\lambda_i^* \lambda_i\right) = 1$$

Eigenvalues of a Unitary Matrix have unity magnitude (cont.)

- Because:

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- Taking the magnitude of each side:

$$\left|\lambda_i^* \lambda_i\right| = 1$$

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- Because:

$$\mathbf{a}_i^\dagger \mathbf{a}_i = \left(\lambda_i^* \lambda_i\right) \mathbf{a}_i^\dagger \mathbf{a}_i$$

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$$\left|\lambda_i^*\right| \left|\lambda_i\right| = 1$$

Eigenvalues of a Unitary Matrix have unity magnitude (cont.)

- Because:

$$\mathbf{a}_i^\dagger \mathbf{a}_i = \left(\lambda_i^* \lambda_i\right) \mathbf{a}_i^\dagger \mathbf{a}_i$$

$$\left(\lambda_i^* \lambda_i\right) \mathbf{a}_i^\dagger \mathbf{a}_i - \mathbf{a}_i^\dagger \mathbf{a}_i = 0$$

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$$\left(\lambda_i^* \lambda_i\right) = 1$$

- Taking the magnitude of each side:

$$\left|\lambda_i^* \lambda_i\right| = 1$$

$$\left|\lambda_i^*\right| \left|\lambda_i\right| = 1$$

$$\therefore \quad \left|\lambda_i^*\right| = \left|\lambda_i\right| = 1 \quad \text{(QED)}$$

Eigenvalues of a Real-valued Unitary Matrix are Real

- Let $\mathbf{a}_i$ represent an eigenvector and λ_i represent the corresponding eigenvalue of $\mathbf{U}$

$$\mathbf{U}\mathbf{a}_i = \lambda_i \mathbf{a}_i$$

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- Multiply both sides by $\mathbf{a}_i$:

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$$\mathbf{a}_i^\dagger \mathbf{U}^\dagger \mathbf{a}_i = \lambda_i^* \mathbf{a}_i^\dagger \mathbf{a}_i$$

$$\mathbf{a}_i^\dagger \left(\mathbf{U}^\dagger \mathbf{a}_i\right) = \lambda_i^* \left(\mathbf{a}_i \bullet \mathbf{a}_i\right)$$

Eigenvalues of a Real-valued Unitary Matrix are Real (cont.)

$$\mathbf{a}_i^\dagger\left(\mathbf{U}^\dagger\mathbf{a}_i\right)=\lambda_i^*\left(\mathbf{a}_i\bullet\mathbf{a}_i\right)$$

- Inner (dot) product of eigenvectors of unitary matrix, $\mathbf{U}$, must be unity valued:

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$$\mathbf{a}_i^\dagger\left(\mathbf{U}^\dagger\mathbf{a}_i\right)=\lambda_i^*\left(1\right)=\lambda_i^*$$

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Eigenvalues of a Real-valued Unitary Matrix are Real (cont.)

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$$\lambda_i=\lambda_i^*$$

∴ real-valued unitary matrix eigenvalues are real (QED)

Eigensystem Facts for Unitary matrix, $\mathbf{U}$

- All Unitary matrices are also normal
  - May or may not be Hermitian
- All unitary matrices are full rank and thus have n distinct nontrivial eigenvectors that are orthogonal
- All eigenvalues are "roots of unity" meaning they lie on the unit circle in the complex plane
  - eigenvalue magnitude is unity
  - for real-valued $\mathbf{U}$, eigenvalues are ± 1

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Spectral Decomposition of Matrix

- Also known as the Eigendecomposition
- Recall that the Spectrum of an $n \times n$ Matrix, $\hat{\mathbf{A}}$, is its *set of eigenvalues*, $\{\lambda_i | i=1,n\}$
- The Spectral Decomposition holds for and Normal matrix, $\hat{\mathbf{A}}\hat{\mathbf{A}}^{\dagger}=\hat{\mathbf{A}}^{\dagger}\hat{\mathbf{A}}$, (includes all Hermitian and Unitary matrices are normal)
- Let $\hat{\mathbf{A}}$ be $n \times n$ Square Matrix with n Linearly Independent Eigenvectors, $\boldsymbol{\lambda}_i$
- Spectral Decomposition allows us to Analyze/Represent Operators based on their Eigensystem

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The Spectral Theorem

- The Spectral Theorem states:

$$\hat{\mathbf{A}} = \boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Lambda}^{-1}$$

- $\boldsymbol{\Lambda}$ is $n \times n$ Matrix with i^{th} Column Vector equivalent to the i^{th} Eigenvector, $\boldsymbol{\lambda}_i$
- $\mathbf{D}$ is a Diagonal Matrix with Corresponding Eigenvalues λ_i to Eigenvectors $\boldsymbol{\lambda}_i$
- Alternatively:

$$\hat{\mathbf{A}} = \sum_{i=1}^{n} \boldsymbol{\lambda}_i \left(\mathbf{a}_i \otimes \mathbf{a}_i^{\dagger} \right) = \sum_{i=1}^{n} \boldsymbol{\lambda}_i \left| a_i \right\rangle \left\langle a_i \right|$$

  - where ⊗ denotes the “outer” or “tensor” or “Kronecker” operation

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Spectral Theorem Example

- Consider an operator, $\mathbf{X}$, (known as the Pauli-$\mathbf{X}$ operator in QM):

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{X}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{X}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Computing the Eigensystem of $\mathbf{X}$ and Applying the Spectral Theorem yields the decomposition: $\mathbf{X} = \boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Lambda}^{-1}$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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Spectral Theorem Example (cont.)

- Spectral Decomposition of $\mathbf{X}$: $\mathbf{X} = \boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Lambda}^{-1}$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- In QIS, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^{-1}$ are "Hadamard" operators, $\mathbf{H}$, and $\mathbf{D}$ is the Pauli-$\mathbf{Z}$ operator, $\mathbf{Z}$ (for $\mathbf{X}$ only)

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{X} = \boldsymbol{\Lambda}\mathbf{D}\boldsymbol{\Lambda}^{-1} = \mathbf{HZH}$$

Matrix Operators

Matrix Operator

- Square Matrices can Represent a Transformation of an Element in a Vector Space to Another Element in that same Space
- When there is a Relationship between the two Vector Elements, Matrices can formed that perform Mapping in Accordance with that Relationship
- If the Relationship is a Mathematical "operation" then such Matrices are called "Operators"
  - sometimes denoted with a "hat" to emphasize they are operators
- Rich set of mathematics in Operator Theory

Matrix Operator Example

- Example: Relationship is to increment each component of a 2D vector, $\mathbf{x}$ yielding vector $\mathbf{x}_{+1}$.

$$\mathbf{x}_{+1} = \hat{\mathbf{A}}\mathbf{x}$$

$$\begin{bmatrix} x_1 + 1 \\ x_2 + 1 \end{bmatrix} = \hat{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \hat{\mathbf{A}} = \begin{bmatrix} 1 & \frac{1}{x_2} \\ \frac{1}{x_1} & 1 \end{bmatrix}$$

- Numerical check: Let $\mathbf{x}^T = [\ 3 \ \ 5\ ]$

$$\mathbf{x}_{+1} = \hat{\mathbf{A}}\mathbf{x}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 1 & \frac{1}{5} \\ \frac{1}{3} & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{5} \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Linear Differential Equation System

- Consider a system of first-order differential equations of the form: $\frac{du_1(t)}{dt} = -u_1(t) + 2u_2(t) \quad \frac{du_2(t)}{dt} = u_1(t) - 2u_2(t)$
- System can be expressed as a matrix equation:

$$\frac{d\mathbf{u}(t)}{dt} = \hat{\mathbf{D}}\mathbf{u}(t) \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \hat{\mathbf{D}} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{du_1(t)}{dt} \\ \frac{du_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

- $\hat{\mathbf{D}}$ is a Differential Matrix Operator

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Linear Differential Equation System

- General solution of this equation: $\frac{d\mathbf{u}(t)}{dt} = \hat{\mathbf{D}}\mathbf{u}(t)$
- Assuming that the $u_i(t)$ functions are in a linear form:

$$\mathbf{u}(t) = e^{\hat{\mathbf{D}}t}$$

- Thus, finding the operator is equivalent to solving the differential equation.
- In this case, $\frac{du_1(t)}{dt} = -u_1(t) + 2u_2(t) \quad \frac{du_2(t)}{dt} = u_1(t) - 2u_2(t)$
- Thus,

$$\hat{\mathbf{D}} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \quad \mathbf{u}(t) = e^{\hat{\mathbf{D}}t} = e^{\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} t}$$

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Linear Differential Equation System

- Find eigensystem of the operator:

$$c(\lambda) = \lambda^2 + 3\lambda = \lambda(\lambda + 3), \quad \lambda_1 = 0, \lambda_1 = -3$$

- The solution has the form:

$$u_i(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- We have the eigenvalues, find eigenvectors that satisfy:

$$\hat{\mathbf{D}}\mathbf{x}_1 = \mathbf{0}$$

$$\hat{\mathbf{D}}\mathbf{x}_2 = -3\mathbf{x}_2 \qquad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Given initial conditions, we can now solve for the constants in the solution.
- Assume: $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Linear Differential Equation System

- With this eigensystem, the solution is of the form:

$$\lambda_1 = 0, \lambda_1 = -3 \qquad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- The solution has the form:

$$\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u_i(0) = c_1 e^{0t} + c_2 e^{-3t}$$

$$\mathbf{u}(0) = c_1(1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{1}{3}$$

$$\mathbf{u}(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Linear Differential Equation System

- Differential equation system and solution:

$$\frac{d\mathbf{u}(t)}{dt} = \hat{\mathbf{D}}\mathbf{u}(t) \quad \hat{\mathbf{D}} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \quad \mathbf{u}(t) = e^{\hat{\mathbf{D}}t}$$

- The solution has the form:

$$\mathbf{u}(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lim_{t\to\infty} \mathbf{u}(t) = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

- Eigenvalues of operator indicate:
  1) Stability, negative eigenvalues, need:

$$\mathbf{u}(t) \to 0, e^{\lambda_i t} \to 0, \mathrm{Re}\left[\boldsymbol{\lambda}_i\right] < 0 \forall i$$

  2) Steady State, $\exists \boldsymbol{\lambda}_i = 0, \text{ and } \mathrm{Re}\left[\boldsymbol{\lambda}_j\right] < 0 \forall \left(j \neq i\right)$
  3) Instability if, $\exists \mathrm{Re}\left[\boldsymbol{\lambda}_i\right] > 0$

Another Example

- Consider the set of scalar functions:

$$x_1 = \cos^2 t \qquad x_2 = \sin^2 t \qquad x_3 = \sin 2t$$

- Let the vectors $\mathbf{x}$ and $\mathbf{y}$ be defined as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad y_i = \frac{dx_i}{dt}$$

- This system can be described with a differential operator matrix as: $\mathbf{y} = \hat{\mathbf{D}}\mathbf{x}$
- Let us determine the operator matrix, $\hat{\mathbf{D}}$

Finding the Operator

- First, we need to determine if we can express the y_i as LINEAR combinations of the x_i.
  - This would allow us to express the solution to the differential equations as linear functions:

$$\mathbf{y} = \hat{\mathbf{D}}\mathbf{x} \qquad \mathbf{y}\left(t\right) = e^{\hat{\mathbf{D}}t}$$

- Recall the chain rule:

$$\frac{df\left(g\left(t\right)\right)}{dt} = \left(\frac{df}{dg}\right)\left(\frac{dg}{dt}\right)$$

- Then:

$$y_1 = \frac{dx_1}{dt} = -2\cos t \sin t = -\sin(2t)$$

Finding the Operator (cont.)

- Then:

$$y_1 = \frac{dx_1}{dt} = -2\cos t \sin t = -\sin(2t)$$

$$y_2 = \frac{dx_2}{dt} = 2\cos t \sin t = \sin(2t)$$

$$y_3 = \frac{dx_3}{dt} = 2\cos(2t) = \cos^2 t - \sin^2 t$$

- Note that:

$$x_1 = \cos^2 t \qquad x_2 = \sin^2 t \qquad x_3 = \sin 2t$$

$$y_1 = -2\cos t \sin t = -\sin(2t) = -x_3$$

$$y_2 = 2\sin t \cos t = \sin(2t) = x_3$$

$$y_3 = 2\cos\left(2t\right) = \cos^2 t - \sin^2 t = x_1 - x_2$$

Finding the Operator (cont.)

- This means that there is the LINEAR relationship we are looking for since:

$$x_1 = \cos^2 t \qquad x_2 = \sin^2 t \qquad x_3 = \sin 2t$$

$$y_1 = -x_3 \qquad y_2 = x_3 \qquad y_3 = x_1 - x_2$$

- Finding these relationships allows to construct the operator as a matrix of constant values:

$$\hat{\mathbf{D}} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{y} = \hat{\mathbf{D}}\mathbf{x} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cos^2 t \\ \sin^2 t \\ \sin(2t) \end{bmatrix} = \begin{bmatrix} -\sin(2t) \\ \sin(2t) \\ \cos^2 t - \sin^2 t \end{bmatrix} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}$$

Finding the Operator (cont.)

- We know that the solution of the differential equations is of the form of an exponential of the operator matrix multiplied by time:

$$\hat{\mathbf{D}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{y} = \hat{\mathbf{D}}\mathbf{x} \qquad \mathbf{y}(t) = e^{\hat{\mathbf{D}}t}$$

- Lets analyze the operator. By inspection, we see that the rank is 2.

$$\hat{\mathbf{D}}^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\widehat{\mathbf{D}}^3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} = (-2)\widehat{\mathbf{D}}$$

Finding the Solution

- The solution of this system is of the form:

$$y_i(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}$$

- Find the eigensystem of the operator.

$$\lambda_1 = 0, \lambda_2 = i\sqrt{2}, \lambda_3 = -i\sqrt{2}$$

- Thus, we see this system has a steady state since:

$$\exists \lambda_i = 0, \text{ and } \operatorname{Re}\left[\lambda_j\right] < 0 \forall \left(j \neq i\right)$$

- The imaginary values indicate that the exponentials associated just revolve around the complex unit circle.
- Eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} i\sqrt{2} \\ -i\sqrt{2} \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -i\sqrt{2} \\ i\sqrt{2} \\ 2 \end{bmatrix}$$

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Finding the Solution (cont.)

- The eigensystem is of the form:

$$y_i(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} i\sqrt{2} \\ -i\sqrt{2} \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -i\sqrt{2} \\ i\sqrt{2} \\ 2 \end{bmatrix}$$

$$\lambda_1 = 0, \lambda_2 = i\sqrt{2}, \lambda_3 = -i\sqrt{2}$$

- We can find an explicit solution given initial conditions.
- Assume:

$$\mathbf{y}\left(t = 0\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{(0)t} + c_2 \begin{bmatrix} -i\sqrt{2} \\ i\sqrt{2} \\ 2 \end{bmatrix} e^{-i\sqrt{2}t} + c_3 \begin{bmatrix} i\sqrt{2} \\ -i\sqrt{2} \\ 2 \end{bmatrix} e^{i\sqrt{2}t}$$

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Finding the Solution (cont.)

- Solve for constants c_1, c_2, c_3 by solving this set o linear equations:

$$\mathbf{y}(0) = \begin{bmatrix}1\\1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\\0\end{bmatrix}e^{(0)t} + c_2\begin{bmatrix}-i\sqrt{2}\\i\sqrt{2}\\2\end{bmatrix}e^{-i\sqrt{2}t} + c_3\begin{bmatrix}i\sqrt{2}\\-i\sqrt{2}\\2\end{bmatrix}e^{i\sqrt{2}t}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\\0\end{bmatrix}(1) + c_2\begin{bmatrix}-i\sqrt{2}\\i\sqrt{2}\\2\end{bmatrix}e^{-i\sqrt{2}t} + c_3\begin{bmatrix}i\sqrt{2}\\-i\sqrt{2}\\2\end{bmatrix}e^{i\sqrt{2}t}$$

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Finding the Solution (cont.)

- These are my solutions for c_1, c_2, c_3 (*they should be double-checked and simplified*):

$$c_1 = 1 + \left(\frac{i\frac{\sqrt{2}}{2} + \left(1 + i2\sqrt{2}\right)e^{-i\sqrt{2}t} - i\sqrt{2}e^{i\sqrt{2}t} - 4}{2 + i\frac{3\sqrt{2}}{2}}\right)$$

$$c_2 = \frac{1 + i2\sqrt{2} - 2e^{i\sqrt{2}t}}{2 + i\frac{3\sqrt{2}}{2}}$$

$$c_3 = \frac{1 - i\sqrt{2}e^{-i\sqrt{2}t}}{2 + i\frac{3\sqrt{2}}{2}}$$

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Finding the Solution (cont.)

- Final Solution for initial conditions of $\mathbf{y}^T = [1 \quad 1 \quad 1]$ is:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \quad \lambda_1 = 0, \lambda_2 = i\sqrt{2}, \lambda_1 = -i\sqrt{2}$$

$$y(t) = c_1 e^{0t} + c_2 e^{i\sqrt{2}t} + c_3 e^{-i\sqrt{2}t} = c_1 + c_2 e^{i\sqrt{2}t} + c_3 e^{-i\sqrt{2}t}$$

$$y(t) = 1 + \left(\frac{i\frac{\sqrt{2}}{2} + \left(1 + i2\sqrt{2}\right)e^{-i\sqrt{2}t} - i\sqrt{2}e^{i\sqrt{2}t} - 4}{2 + i\frac{3\sqrt{2}}{2}} \right)$$

$$+ \left(\frac{1 + i2\sqrt{2} - 2e^{i\sqrt{2}t}}{2 + i\frac{3\sqrt{2}}{2}} \right) e^{i\sqrt{2}t} + \left(\frac{1 - i\sqrt{2}e^{-i\sqrt{2}t}}{2 + i\frac{3\sqrt{2}}{2}} \right) e^{-i\sqrt{2}t}$$

Solution contains Oscillating Functions, Sine and Cosine Functions with Imaginary Components!