Mixed Quantum States

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Pure versus Mixed States

- We let a quantum state vector represent a qubit ($n=2$) or qudit ($n>2$)
- We have learned about quantum superposition and Born's rule
- The projective measurement of a quantum state with respect to a measurement basis leads to a *subjective probability*
- These are ***Pure States***
- However, it is possible that we have *missing information regarding the quantum state of a system* is comprised of an ensemble of different pure states: a ***Mixed State***
- This is an *objective probability* and is thus a probability in the classical sense
- Mixed State observation yields one of the state vectors in the ensemble with a "Classical Probability" and NOT a quantum probability
- Indistinguishable (experimentally) from observation of a pure state

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Mixed Quantum State

- The collection (ensemble) of pure states is truly a random selection of different pure states
  - each of which can be in a state of superposition
- Name comes from "Quantum Statistical Mechanics"
  - quantum system comprised of two quantum subsystems that could be in different pure states
  - could be a quantum system that was initialized at an unknown state, thus the set of evolved states a later time could be one of several with a distinct (objective) probability
- Mixed states are properties of the system and NOT the result of the QM postulates
- Since energy is conserved, the norm of each vector in the ensemble comprising a mixed state is LESS THAN UNITY
  - Bloch Sphere: they are "inside" the Bloch sphere

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Density Operator Definition

- Mixed State is Described by Density Operator, ρ
  - describes a statistical ensemble of systems, or in QIS, a statistical ensemble of possible quantum states of a system
- Could use Density Operators in place of quantum state kets (vectors)
  - practice is to use them only for mixed states
- Quantum state density operator is computed for a system of n states as:

$$\rho = \sum_{i=1}^{n} p_i |\Psi_i\rangle\langle\Psi_i|$$

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Quantum Density Operator

$$\rho = \sum_{i=1}^{n} p_i |\Psi_i\rangle\langle\Psi_i|$$

- The p_i values are non-negative real values that are the probability the system is pure state i

$$p_i = \text{Prob}\left[\text{QS-state} = |\Psi_i\rangle\right]$$

- Consider QS in a single Pure State:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- Compute the density matrix: $n = 1, \; p_1 = 1$

$$\rho = \sum_{i=1}^{1}(1)|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{bmatrix}$$

- Observe the trace of the density matrix:

$$\text{Tr}[\rho] = \text{Tr}\begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{bmatrix} = \alpha\alpha^* + \beta\beta^* = 1$$

Born's Rule!!!

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Quantum Density Operator

- Consider QS in a one of two States:

$$n = 2, \; (p_1, p_2) \qquad |\Psi_1\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\Psi_2\rangle = \delta|0\rangle + \gamma|1\rangle$$

- Compute the density matrix:

$$\rho = \sum_{i=1}^{2} p_i |\Psi_i\rangle\langle\Psi_i| = |\Psi_1\rangle\langle\Psi_1| = p_1\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} + p_2\begin{bmatrix} \delta \\ \gamma \end{bmatrix}\begin{bmatrix} \delta^* & \gamma^* \end{bmatrix}$$

$$= p_1\begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{bmatrix} + p_2\begin{bmatrix} \delta\delta^* & \delta\gamma^* \\ \delta^*\gamma & \gamma\gamma^* \end{bmatrix} = \begin{bmatrix} p_1\alpha\alpha^* & p_1\alpha\beta^* \\ p_1\alpha^*\beta & p_1\beta\beta^* \end{bmatrix} + \begin{bmatrix} p_2\delta\delta^* & p_2\delta\gamma^* \\ p_2\delta^*\gamma & p_2\gamma\gamma^* \end{bmatrix}$$

$$= \begin{bmatrix} (p_1\alpha\alpha^* + p_2\delta\delta^*) & (p_1\alpha\beta^* + p_2\delta\gamma^*) \\ (p_1\alpha^*\beta + p_2\delta^*\gamma) & (p_1\beta\beta^* + p_2\gamma\gamma^*) \end{bmatrix}$$

- Observe the trace of the density matrix:

$$\text{Tr}[\rho] = \text{Tr}\begin{bmatrix} (p_1\alpha\alpha^* + p_2\delta\delta^*) & (p_1\alpha\beta^* + p_2\delta\gamma^*) \\ (p_1\alpha^*\beta + p_2\delta^*\gamma) & (p_1\beta\beta^* + p_2\gamma\gamma^*) \end{bmatrix} = p_1\alpha\alpha^* + p_2\delta\delta^* + p_1\beta\beta^* + p_2\gamma\gamma^*$$

$$= p_1(\alpha\alpha^* + \beta\beta^*) + p_2(\delta\delta^* + \gamma\gamma^*) = p_1(1) + p_2(1) = p_1 + p_2 = 1$$

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Quantum Density Operator

- Consider QS in one of an ensemble of n states:

$$n,\ \left(p_1, p_2, \cdots, p_n\right) \qquad \left|\Psi_i\right\rangle = \alpha_i\left|0\right\rangle + \beta_i\left|1\right\rangle$$

- Compute the density matrix:

$$\rho = \sum_{i=1}^{n} p_i \left|\Psi_i\right\rangle\left\langle\Psi_i\right| = \sum_{i=1}^{n} p_i \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}\begin{bmatrix} \alpha_i^* & \beta_i^* \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} p_i\alpha_i\alpha_i^* & \sum_{i=1}^{n} p_i\alpha_i\beta_i^* \\ \sum_{i=1}^{n} p_i\alpha_i^*\beta_i & \sum_{i=1}^{n} p_i\beta_i\beta_i^* \end{bmatrix}$$

- Observe the trace of the density matrix:

$$\mathrm{Tr}\begin{bmatrix} \sum_{i=1}^{n} p_i\alpha_i\alpha_i^* & \sum_{i=1}^{n} p_i\alpha_i\beta_i^* \\ \sum_{i=1}^{n} p_i\alpha_i^*\beta_i & \sum_{i=1}^{n} p_i\beta_i\beta_i^* \end{bmatrix} = \sum_{i=1}^{n} p_i\alpha_i\alpha_i^* + \sum_{i=1}^{n} p_i\beta_i\beta_i^* = \sum_{i=1}^{n}\left(p_i\alpha_i\alpha_i^* + p_i\beta_i\beta_i^*\right)$$

$$= \sum_{i=1}^{n} p_i\left(\alpha_i\alpha_i^* + \beta_i\beta_i^*\right) = \sum_{i=1}^{n} p_i\left(1\right) = \sum_{i=1}^{n} p_i = 1$$

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Example using Photon Polarization

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Photon Polarization:
Pure and Mixed States

https://www.youtube.com/watch?v=bP-bBVfmzEw (9:09)

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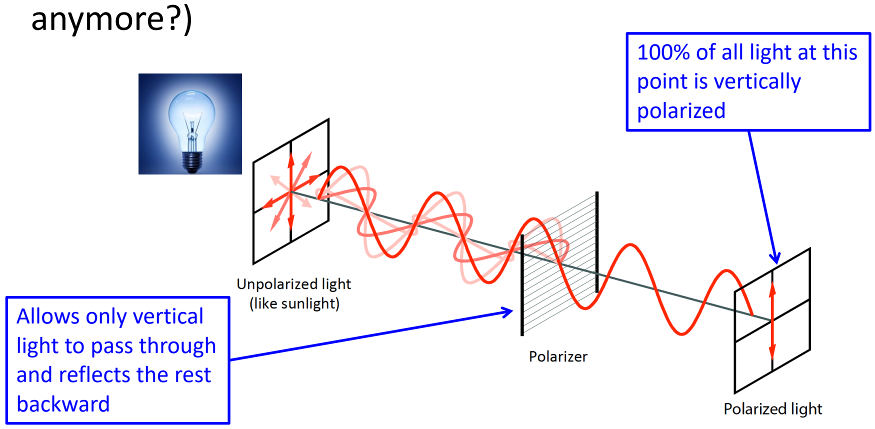
Light Polarization Example

- Consider a System where incoherent light is filtered to provide only vertically polarized light
- The light is generated by an incoherent source like the sun or an incandescent light bulb (is there such a thing anymore?)



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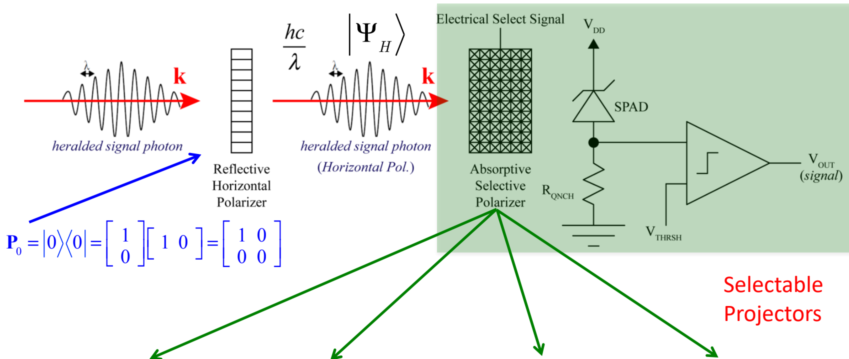
Selectable-Pol. Detector

- Assume the Energy detector is equipped with its own polarizing filter that can be set to one or more of:

$$\{|H\rangle, |V\rangle, |+\rangle, |-\rangle\} = \{|\leftrightarrow\rangle, |\updownarrow\rangle, |\nearrow\rangle, |\nwarrow\rangle\}$$



$$\mathbf{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{P}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{P}_{\nearrow} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{P}_{\nwarrow} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

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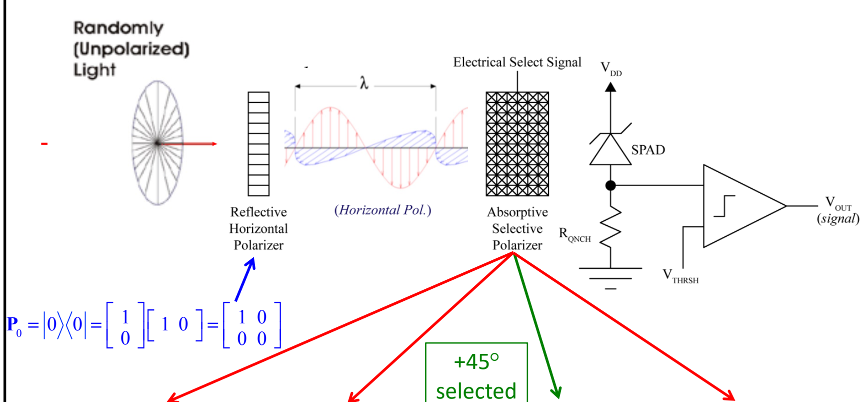
Selectable-Pol. Detector

- Assume selectable polarizer is set to +45° only with an incident *EM* Horizontally-polarized wave being incident (classical):



$$\mathbf{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{P}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{P}_{\nearrow} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{P}_{\nwarrow} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

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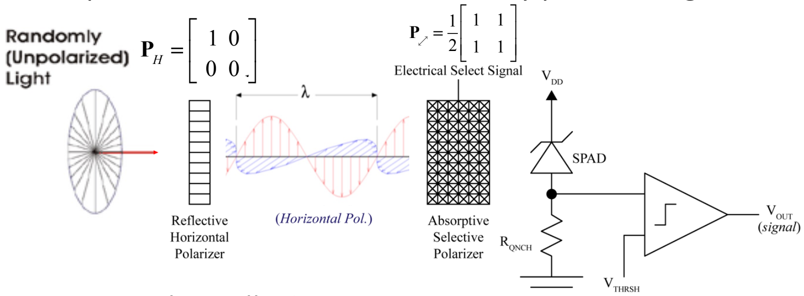
Classic Detection

- Classically, the output of the detector will output a measurement that is proportional to a smaller amount of amplitude of the incident horizontally polarized light



- How much smaller?

- What happens to the rest of the energy in the incident horizontally polarized wave?

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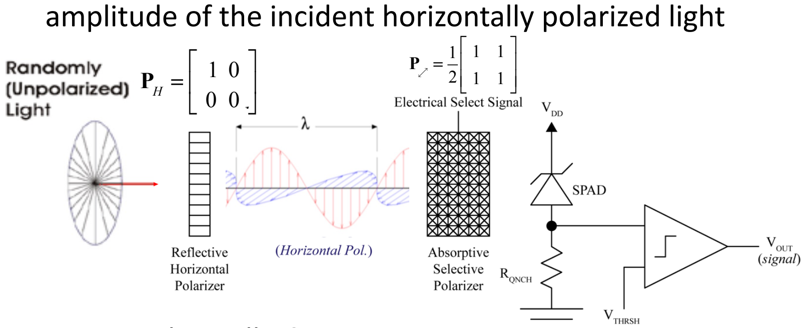
Classic Detection

- Classically, the output of the detector will output a measurement that is proportional to a smaller amount of amplitude of the incident horizontally polarized light



- How much smaller?
  Amplitude at output of sel. polarizer is decreased by factor of cos(45°).
- What happens to the rest of the energy in the incident horizontally polarized wave?

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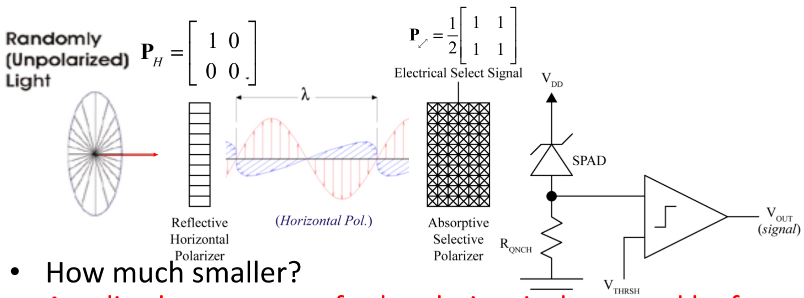
Classic Detection

- Classically, the output of the detector will output a measurement that is proportional to a smaller amount of amplitude of the incident horz. polarized light



- How much smaller?
  Amplitude at output of sel. polarizer is decreased by factor of cos(45°).
- What happens to the rest of the energy in the incident horizontally polarized wave? Absorbed/converted to heat (resistive current dissipation)

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Projected Polarization Energy

- Some Energy from the Horizontally Polarized Wave WILL Pass through the +45° Slant Polarizer
- Some Energy will be Absorbed and converted to heat
  - illustrated on unit circle with axes representing polarization
- Incident wave E-field amplitude: A_I (V/m)
- Measured E-field:

$$A_M = A_I \cos\left(\frac{\pi}{4}\right) \text{ (V/m)} \quad P_{A_M} = A_I^2 \cos^2\left(\frac{\pi}{4}\right) \text{ (W/m}^2\text{)}$$

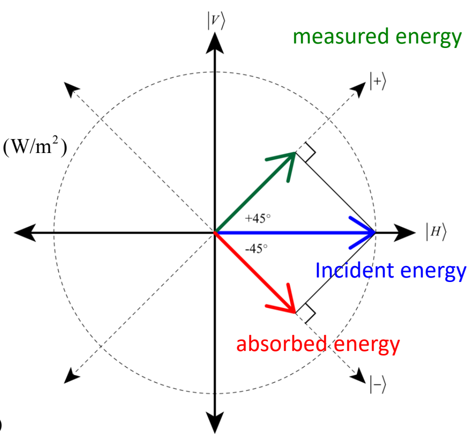
- Absorbed E-field:

$$A_A = A_I \cos\left(\frac{-\pi}{4}\right) \text{ (V/m)}$$

$$P_{A_A} = A_I^2 \cos^2\left(\frac{-\pi}{4}\right) \text{ (W/m}^2\text{)}$$

- Pol. Loss:

$$L_{POL} = 10\log_{10}\left[\cos^2\left(\frac{-\pi}{4}\right)\right] = 10\log_{10}\left(\frac{1}{2}\right) = -3 \text{ (dB)}$$



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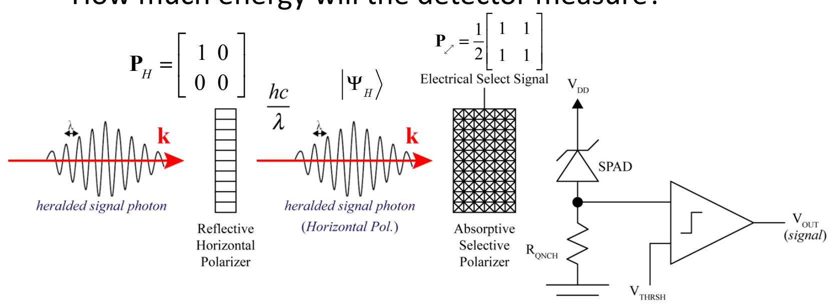
Single Photon Source

- The light is generated by a very, very weak source only emits a single photon at a time
- How much energy will the detector measure?



- How much smaller?
- What happens to the energy in the incident photon?

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Is this accurate for single photon?

- Some Energy from the Horizontally Polarized Photon WILL Pass through the $+45°$ Slant Polarizer
- Some Energy will be Absorbed and converted to heat
  - illustrated on unit circle with axes representing polarization
- Incident wave E-field amplitude: A_I (V/m)
- Measured E-field:

$$A_M = A_I \cos\left(\frac{\pi}{4}\right) \text{ (V/m)} \quad P_{A_M} = A_I^2 \cos^2\left(\frac{\pi}{4}\right) \text{ (W/m}^2\text{)}$$

- Absorbed E-field:

$$A_A = A_I \cos\left(\frac{-\pi}{4}\right) \text{ (V/m)}$$

$$P_{A_A} = A_I^2 \cos^2\left(\frac{-\pi}{4}\right) \text{ (W/m}^2\text{)}$$

- Pol. Loss:

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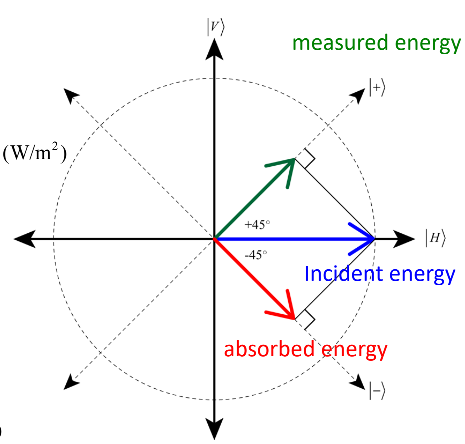

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Is this accurate for single photon?

- **NO! Impossible to divide a single quantum of energy (photon)**
- **We must determine the observable and use quantum measurement theory in this case**
- Incident wave E-field amplitude: A_I (V/m)
- Measured E-field:

$$A_M = A_I \cos\left(\frac{\pi}{4}\right) \text{ (V/m)} \quad P_{A_M} = A_I^2 \cos^2\left(\frac{\pi}{4}\right) \text{ (W/m}^2\text{)}$$

- Absorbed E-field:

$$A_A = A_I \cos\left(\frac{-\pi}{4}\right) \text{ (V/m)}$$

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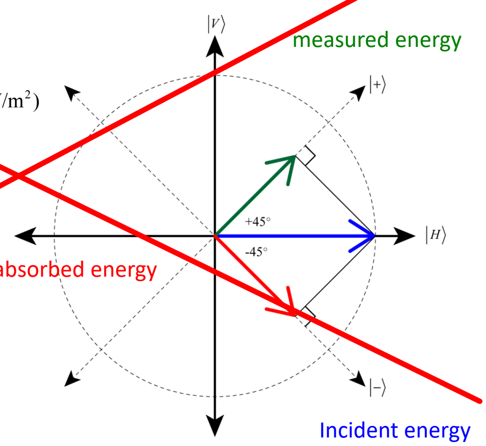

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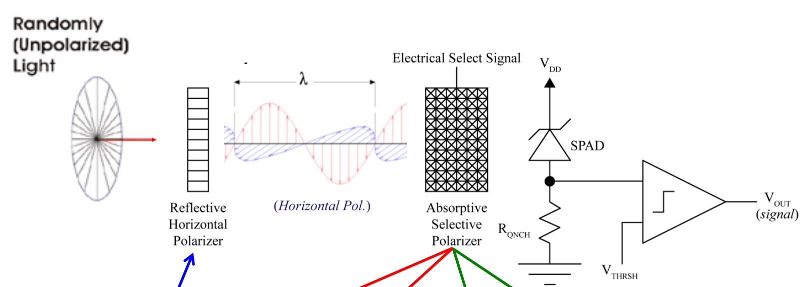
Selectable-Pol. Detector

- Assume selectable polarizer is *set to both +45° and -45°* with EM Horizontally-polarized incident wave (classical):



$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

+45° selected

-45° selected

$$\mathbf{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{P}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{P}_{\nearrow} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{P}_{\nwarrow} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

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Projected Polarization Energy

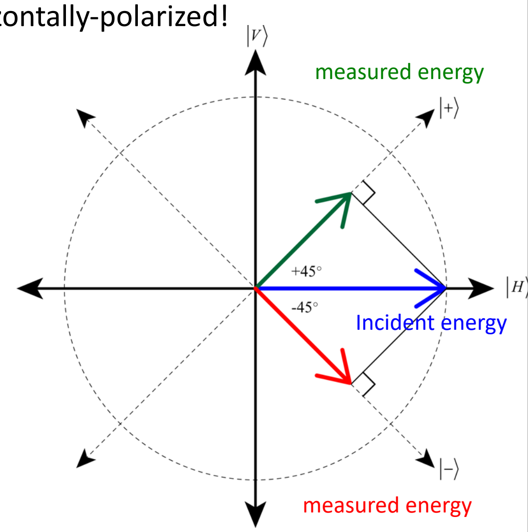
- All Energy from the Horizontally Polarized Wave WILL Pass through the +/-45° Slant Polarizer (ideally)
- Even though E-field is Horizontally-polarized!
- Incident wave E-field amplitude: A_I (V/m)
- Measured E-field:

$$A_M = A_I \cos\left(\frac{\pi}{4}\right) + A_I \cos\left(\frac{-\pi}{4}\right) \text{(V/m)}$$

$$P_{A_M} = A_I^2\left[\cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{-\pi}{4}\right)\right]$$

$$= A_I^2\left[\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right] = A_I^2 \text{(W/m}^2\text{)}$$

- No polarization loss!!!!



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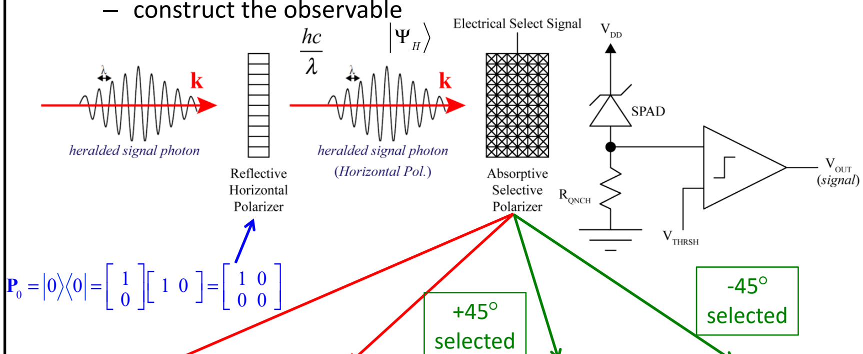
Selectable-Pol. Detector

- Assume selectable polarizer is set to both +45° and -45° with single photon source (quantum):
- We must use quantum measurement theory
  - construct the observable



$$\mathbf{P}_H = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \quad \mathbf{P}_V = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \quad \mathbf{P}_{\nearrow} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \quad \mathbf{P}_{\nwarrow} = \frac{1}{2}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$$

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Photon Polarization

- The light is generated by a very, very weak source only emits a single photon at a time
- The observable of interest for the photon is its slant-45 polarization state:
- Assume the incident photon is horizontally polarized due to the reflective linear horizontal polarizer
- The Observable is (from our study of Measurements)
  - measurement outcomes (eigenvalues), $\{\lambda_{\nearrow}, \lambda_{\nwarrow}\}=\{+1,-1\}$

$$\mathbf{A}_{\mathbf{X}} = \lambda_{\nearrow}|\nearrow\rangle\langle\nearrow| + \lambda_{\nwarrow}|\nwarrow\rangle\langle\nwarrow| = \lambda_{\nearrow}\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \lambda_{\nwarrow}\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}$$

$$= \lambda_{\nearrow}\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \lambda_{\nwarrow}\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = (1)\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + (-1)\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

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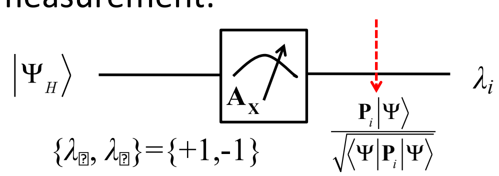
Photon Polarization (cont.)

- Projective measurement:



- The expected value for the horizontally-polarized photon when measured with the slant polarization observable is:

$$\langle\mathbf{A}_{\mathbf{X}}\rangle = \langle\boldsymbol{\Psi}_H|\mathbf{A}_{\mathbf{X}}|\boldsymbol{\Psi}_H\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

- The probability that the measurement outcome is $\lambda_{\nearrow}=+1$:

$$\left\|\mathbf{P}_{\nearrow}|\boldsymbol{\Psi}_H\rangle\right\|^2 = \left\|\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right\|^2 = \left\|\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\right\|^2 = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{2}$$

- The probability that the measurement outcome is $\lambda_{\nwarrow}=-1$:

$$\left\|\mathbf{P}_{\nwarrow}|\boldsymbol{\Psi}_H\rangle\right\|^2 = \left\|\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right\|^2 = \left\|\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right\|^2 = \left|\frac{1}{2}\right|^2 + \left|\frac{-1}{2}\right|^2 = \frac{1}{2}$$

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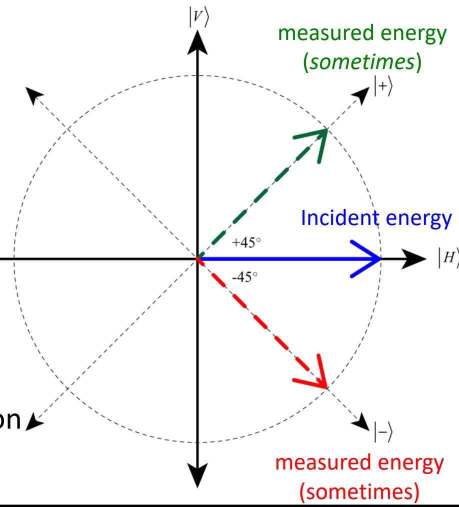
Projected Polarization Energy

- All Energy from the Horizontally Polarized Wave WILL Pass through the +/-45° Slant Polarizer (ideally)
- Even though *Photon* is Horizontally-polarized!
- Photon is an *EM* quantum and cannot be divided
- Detector measures each incident photon as either either +45° or -45° polarized state (ideally)
- No polarization loss!!!!
- Classic case is an enormous number of photons, thus energy appears to be divided since half of the photons detected with +45° and the other half with -45° polarization



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Interpretation

- The incident horizontally polarized photon, $|\Psi_H\rangle$ was Observed using the slant-45 measurement basis
- Thus, the photon while being in a basis state with respect to the horizontal/vertical polarization basis, is in a state of perfect superposition with respect to the slant-45 basis:

$$|\Psi_H\rangle = |H\rangle = \frac{|\nearrow\rangle + \langle\nwarrow|}{\sqrt{2}}$$

- This probabilistic behavior of being either +45° or -45° polarized is due to Born's rule, a QM postulate
- This is a Subjective Probability

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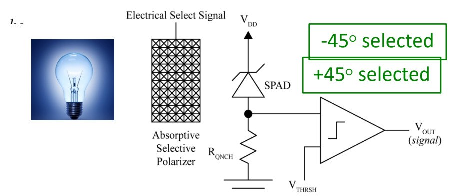
Unpolarized Light

- Unpolarized light source emits a collection of photons each with a *statistical ensemble of all possible polarizations*
  - Each photon cannot be described as a single pure state
- Must be described with a Density Matrix
  - true if LHC/RHC or +45°/-45° polarizer is used as well
- Experiment below yields same outcome as if a polarized photon is incident, but it is NOT due to superposition/projection, due to statistical ensemble of different polarization states
- Cannot distinguish between the 2 cases !!!!!



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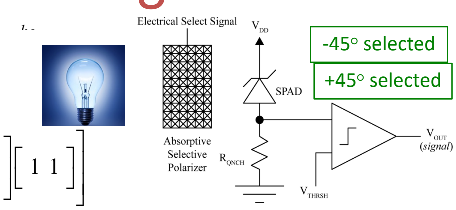
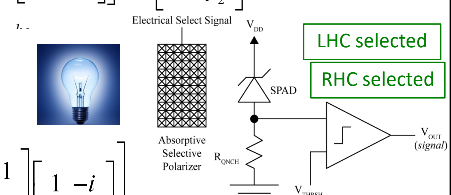
Unpolarized Light



$$\rho = p_1|\Psi_{\nearrow}\rangle\langle\Psi_{\nearrow}| + p_{\nwarrow}|\Psi_{\nwarrow}\rangle\langle\Psi_{\nwarrow}| = p_1\left[\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\right]$$

$$+\,p_2\left[\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}\right] = p_1\left[\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right] + p_2\left[\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix}p_1 & 0\\0 & p_2\end{bmatrix}$$



$$\rho = p_1|\Psi_{\circlearrowleft}\rangle\langle\Psi_{\circlearrowleft}| + p_2|\Psi_{\circlearrowright}\rangle\langle\Psi_{\circlearrowright}| = p_1\left[\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix}\right]$$

$$+\,p_2\left[\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix}\right] = p_1\left[\frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix}\right] + p_2\left[\frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix}p_1 & 0\\0 & p_2\end{bmatrix}$$

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