ECE 5/7383 Introduction to Quantum Informatics

Homework 3 - SOLUTIONS

Complex Amplitudes, the Schrödinger Equation, Projective Measurement, and Information

Assigned: Wednesday, October 7, 2026 Due: Monday, Oct. 19, 2026 at 2:00PM (CDT)

Work every problem by hand and show the intermediate steps; an answer with no supporting work receives no credit. Parts labeled *(for 7383 students only)* are required of ECE 7383 students and will receive NO CREDIT if answered by ECE 5383 students. This homework is worth 100 points total for both ECE 5383 and ECE 7383 students.

Dirac's "bra-ket" notation is used throughout. A ket $|\psi\rangle$ denotes a column vector, the bra $\langle\psi|$ denotes its adjoint, which is a row vector, $\langle\varphi|\psi\rangle$ denotes the inner product, and $|\psi\rangle\langle\varphi|$ denotes the outer product, which is a matrix. The computational basis is $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$, and the Hadamard basis is $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The Pauli operators are

$$\mathbf{X} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$$

A circumflex such as $\hat{\mathbf{H}}$ or $\hat{p}_x$ denotes an operator. The constant e is Euler's number, i satisfies $i^2 = -1$, and z^* denotes the complex conjugate of z. For a complex number z, $|z|$ is its modulus and $\arg(z)$ is its argument, the phase angle measured counterclockwise from the positive real axis, so that $z = |z|e^{i\arg(z)}$. A state $|\psi\rangle$ is normalized when $\langle\psi|\psi\rangle = 1$. Unless a problem states otherwise, use the following values for the physical constants:

$$h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s}, \qquad \hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34}\ \text{J}\cdot\text{s},$$

$$m_e = 9.109 \times 10^{-31}\ \text{kg}, \qquad 1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}.$$

You may submit your completed assignment to me electronically via an email attachment in the form of a PDF file (this is preferable), or you can turn in hardcopy at the beginning of class (2PM) of the due date. The email time and date stamp will serve as proof of your submission date and time for electronic submissions. Grading penalties for late assignments are in force. The grading policy for late assignments is posted on the class webpage/syllabus.

This homework is for your benefit and will help you to prepare for exams, so I hope you take it seriously.

1. **(Algebra of complex amplitudes — 16 pts for 5383; 15 pts for 7383)** Every amplitude in quantum informatics is a complex number, so fluency with complex arithmetic is a prerequisite for everything that follows. Let $z_1 = 3 + 4i$ and $z_2 = 1 - i$.

a) Express each of $z_1 + z_2$, $z_1 z_2$, and z_1/z_2 in the rectangular form $a + ib$ with a and b real. For the quotient, show the step in which you multiply by the conjugate of the denominator.

SOLUTION: Addition is componentwise:

$$z_1 + z_2 = (3 + 1) + (4 - 1)i = 4 + 3i.$$

Multiplication expands term by term, using $i^2 = -1$:

$$z_1 z_2 = (3 + 4i)(1 - i) = 3 - 3i + 4i - 4i^2 = 3 + i + 4 = 7 + i.$$

For the quotient, multiply numerator and denominator by $z_2^* = 1 + i$, which makes the denominator real because $z_2 z_2^* = |z_2|^2$:

$$\frac{z_1}{z_2} = \frac{3 + 4i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{(3 + 4i)(1 + i)}{1^2 + 1^2} = \frac{3 + 3i + 4i - 4}{2} = \frac{-1 + 7i}{2} = -\frac{1}{2} + \frac{7}{2}i.$$

b) Give the modulus and the argument of z_1 and of z_2, and write each in the polar form $|z|e^{i\arg(z)}$. Then verify, using your answer from part (a), that $|z_1 z_2| = |z_1||z_2|$ and that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

SOLUTION: The modulus is $\sqrt{a^2 + b^2}$ and the argument is the angle whose tangent is b/a, taken in the quadrant of the point:

$$|z_1| = \sqrt{3^2 + 4^2} = 5, \qquad \arg(z_1) = \arctan\left(\frac{4}{3}\right) \approx 53.13°, \qquad z_1 = 5e^{i\arctan(4/3)}.$$

$$|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \qquad \arg(z_2) = -\frac{\pi}{4} = -45°, \qquad z_2 = \sqrt{2}\, e^{-i\pi/4}.$$

Note that z_2 lies in the fourth quadrant, since its real part is positive and its imaginary part is negative, so the argument is negative.

From part (a), $z_1 z_2 = 7 + i$, so

$$|z_1 z_2| = \sqrt{49 + 1} = \sqrt{50} = 5\sqrt{2} = |z_1||z_2|. \checkmark$$

$$\arg(z_1 z_2) = \arctan\left(\frac{1}{7}\right) \approx 8.13° = 53.13° - 45° = \arg(z_1) + \arg(z_2). \checkmark$$

c) Let $z = a + ib$ with a and b real. Show by direct expansion that $z + z^* = 2\,\mathrm{Re}(z)$ and that $zz^* = |z|^2$. Then use the second identity to show that $\langle\psi|\psi\rangle$ is real and non-negative for every single-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex.

SOLUTION: With $z = a + ib$ the conjugate is $z^* = a - ib$, so the imaginary parts cancel in the sum and the cross terms cancel in the product:

$$z + z^* = (a + ib) + (a - ib) = 2a = 2\,\mathrm{Re}(z).$$

$$zz^* = (a + ib)(a - ib) = a^2 - iab + iab - i^2 b^2 = a^2 + b^2 = |z|^2.$$

For the state, the bra is $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$. Using orthonormality of the computational basis, meaning $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$,

$$\langle\psi|\psi\rangle = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2.$$

By the identity just proved, each term is a sum of two real squares, so each is real and non-negative, and therefore so is the total. It equals zero only when $\alpha = \beta = 0$, which is the zero vector and not a physical state. This is what allows $|\alpha|^2$ and $|\beta|^2$ to be interpreted as probabilities once the state is normalized.

d) *(for 7383 students only)* Show, by writing $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ and expanding both sides, that $(z_1 z_2)^* = z_1^* z_2^*$. Then use that result to prove that $\langle\varphi|\psi\rangle^* = \langle\psi|\varphi\rangle$ for any two single-qubit states.

SOLUTION: Expanding the product first and then conjugating,

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2), \qquad (z_1 z_2)^* = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2).$$

Conjugating first and then multiplying,

$$z_1^* z_2^* = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2).$$

The two expressions agree term for term, so $(z_1 z_2)^* = z_1^* z_2^*$.

Now let $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$ and $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then $\langle\varphi|\psi\rangle = \gamma^*\alpha + \delta^*\beta$. Conjugating, and using the fact that conjugation distributes over addition and over products,

$$\langle\varphi|\psi\rangle^* = (\gamma^*\alpha)^* + (\delta^*\beta)^* = \gamma\alpha^* + \delta\beta^* = \alpha^*\gamma + \beta^*\delta = \langle\psi|\varphi\rangle.$$

This is the conjugate-symmetry property of the inner product. It is the reason measurement probabilities such as $|\langle\varphi|\psi\rangle|^2$ do not depend on which of the two states is written as the bra.

2. **(The one-dimensional Schrödinger equation for a free particle — 17 pts for 5383; 16 pts for 7383)** The time-independent Schrödinger equation in one dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x).$$

A free particle is one for which the potential energy is constant everywhere, and the constant may be taken to be zero, so $V(x) = 0$ for all x.

a) Show by direct substitution that $\psi(x) = Ae^{ikx}$, with A and k constants, satisfies the free-particle equation, and determine the energy E in terms of $\hbar$, m, and k.

SOLUTION: Differentiating twice, each derivative brings down a factor of ik:

$$\frac{d\psi}{dx} = ikAe^{ikx}, \qquad \frac{d^2\psi}{dx^2} = (ik)^2Ae^{ikx} = -k^2Ae^{ikx} = -k^2\psi(x).$$

Substituting into the equation with $V(x) = 0$,

$$-\frac{\hbar^2}{2m}\left(-k^2\psi(x)\right) = \frac{\hbar^2k^2}{2m}\psi(x) = E\,\psi(x).$$

The equation is satisfied for any A, and matching the coefficient of $\psi(x)$ gives

$$E = \frac{\hbar^2k^2}{2m}.$$

Note that E is not quantized here. Any real k is allowed, because there are no boundary conditions to impose. This is the contrast with the particle in a box, where the walls force $ka = n\pi$.

b) The momentum operator is $\hat{p}_x = -i\hbar\frac{d}{dx}$. Show that $\psi(x) = Ae^{ikx}$ is also an eigenfunction of $\hat{p}_x$ and give its momentum eigenvalue. Then show that your answer to part (a) is consistent with the classical relation $E = p^2/2m$, and state the relationship between k and the de Broglie wavelength λ.

SOLUTION: Applying the momentum operator,

$$\hat{p}_x\psi = -i\hbar\frac{d}{dx}\left(Ae^{ikx}\right) = -i\hbar(ik)Ae^{ikx} = \hbar k\,Ae^{ikx} = \hbar k\,\psi(x).$$

So ψ is an eigenfunction of $\hat{p}_x$ with the real eigenvalue $p = \hbar k$. A free particle in this state therefore has a sharply defined momentum.

Substituting that momentum into the classical kinetic energy,

$$\frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} = \frac{\hbar^2k^2}{2m} = E,$$

which agrees with part (a). Finally, since $p = h/\lambda$ by the de Broglie relation and $p = \hbar k = hk/2\pi$,

$$\frac{h}{\lambda} = \frac{hk}{2\pi} \Rightarrow k = \frac{2\pi}{\lambda}.$$

The constant k is therefore exactly the wavenumber of the matter wave, the same quantity used for classical waves in Homework 2.

c) An electron is in the free-particle state above with $k = 1.00 \times 10^{10}$ m^{-1}. Compute its momentum, its energy in joules and in eV, and its de Broglie wavelength in nm.

SOLUTION:

$$p = \hbar k = (1.0546 \times 10^{-34})(1.00 \times 10^{10}) = 1.0546 \times 10^{-24} \text{ kg} \cdot \text{m/s}.$$

$$E = \frac{p^2}{2m_e} = \frac{(1.0546 \times 10^{-24})^2}{2(9.109 \times 10^{-31})} = \frac{1.1122 \times 10^{-48}}{1.8218 \times 10^{-30}} = 6.105 \times 10^{-19} \text{ J}.$$

$$E = \frac{6.105 \times 10^{-19}}{1.602 \times 10^{-19}} = 3.811 \text{ eV}.$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{1.00 \times 10^{10}} = 6.283 \times 10^{-10} \text{ m} = 0.6283 \text{ nm}.$$

d) Compute the probability density $|\psi(x)|^2$ for this state and describe in one sentence what it says about where the particle is located.

SOLUTION: Using $\left|e^{i\theta}\right| = 1$ for real θ,

$$|\psi(x)|^2 = \psi^* \psi = A^* e^{-ikx} A e^{ikx} = |A|^2 e^0 = |A|^2.$$

The probability density is the same constant at every position, so the particle is equally likely to be found anywhere and its position carries no information at all.

e) *(for 7383 students only)* Explain why this wave function cannot be normalized over the whole real line, and state what that implies about whether a particle can have an exactly defined momentum in practice.

SOLUTION: Normalization would require

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} |A|^2 \, dx = |A|^2 \lim_{L \to \infty} \int_{-L}^{L} d\,x = |A|^2 \lim_{L \to \infty} 2L,$$

which diverges for every $A \neq 0$, and the choice $A = 0$ gives the zero function rather than a state. The plane wave is therefore an idealization rather than a physically realizable state.

The implication is that no particle can have an exactly defined momentum. A physical particle is described by a superposition of plane waves over a range of k values, which forms a localized wave packet that can be normalized. Narrowing the spread in momentum widens the packet in position, which is the content of the uncertainty relation $\Delta x \, \Delta p_x \geq \hbar/2$ that follows from $[\hat{x}, \hat{p}_x] = i\hbar$.

3. **(Projective measurement in two different bases — 18 pts for 5383; 17 pts for 7383)** A qubit is prepared in the state

$$|\psi\rangle = \frac{1+2i}{\sqrt{14}}|0\rangle + \frac{3}{\sqrt{14}}|1\rangle.$$

a) Verify that $|\psi\rangle$ is normalized. Then write the computational-basis projectors $\widehat{\mathbf{P}}_0 = |0\rangle\langle 0|$ and $\widehat{\mathbf{P}}_1 = |1\rangle\langle 1|$ as 2×2 matrices, and verify the three defining properties $\widehat{\mathbf{P}}_0 + \widehat{\mathbf{P}}_1 = \mathbf{I}$, $\widehat{\mathbf{P}}_j^2 = \widehat{\mathbf{P}}_j$, and $\widehat{\mathbf{P}}_0\widehat{\mathbf{P}}_1 = \mathbf{0}$.

SOLUTION: The squared modulus of each amplitude is taken using $|a+ib|^2 = a^2 + b^2$:

$$\langle\psi|\psi\rangle = \frac{|1+2i|^2}{14} + \frac{|3|^2}{14} = \frac{1+4}{14} + \frac{9}{14} = \frac{5+9}{14} = 1. \checkmark$$

The projectors are outer products of the basis kets with their own bras:

$$\widehat{\mathbf{P}}_0 = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \qquad \widehat{\mathbf{P}}_1 = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

$$\widehat{\mathbf{P}}_0 + \widehat{\mathbf{P}}_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{I} \checkmark \; \widehat{\mathbf{P}}_0^2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \widehat{\mathbf{P}}_0 \checkmark \; \widehat{\mathbf{P}}_0\widehat{\mathbf{P}}_1 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \mathbf{0} \checkmark$$

The same computation gives $\widehat{\mathbf{P}}_1^2 = \widehat{\mathbf{P}}_1$. Completeness guarantees the probabilities sum to one, idempotence says that measuring twice in a row gives the same answer, and orthogonality says the two outcomes are mutually exclusive.

b) Compute the probabilities $P(0) = \langle\psi|\widehat{\mathbf{P}}_0|\psi\rangle$ and $P(1) = \langle\psi|\widehat{\mathbf{P}}_1|\psi\rangle$ of the two outcomes of a computational-basis measurement, and verify that they sum to one. Give the normalized post-measurement state when the outcome is 0.

SOLUTION: Writing $|\psi\rangle$ as a column vector and applying the projector,

$$P(0) = \langle\psi|\widehat{\mathbf{P}}_0|\psi\rangle = |\langle 0|\psi\rangle|^2 = \left|\frac{1+2i}{\sqrt{14}}\right|^2 = \frac{5}{14},$$

$$P(1) = |\langle 1|\psi\rangle|^2 = \left|\frac{3}{\sqrt{14}}\right|^2 = \frac{9}{14}, P(0) + P(1) = \frac{5+9}{14} = 1. \checkmark$$

The projection postulate gives the post-measurement state as the projected vector renormalized:

$$\frac{\widehat{\mathbf{P}}_0|\psi\rangle}{\sqrt{P(0)}} = \frac{\frac{1+2i}{\sqrt{14}}|0\rangle}{\sqrt{5/14}} = \frac{1+2i}{\sqrt{5}}|0\rangle.$$

The prefactor has modulus $\sqrt{5}/\sqrt{5} = 1$, so it is a global phase and the post-measurement state is $|0\rangle$ for all physical purposes. This is the collapse of the state onto the eigenvector associated with the observed outcome.

c) Now consider measuring the same prepared state $|\psi\rangle$ in the Hadamard basis instead. Write the projectors $\widehat{\mathbf{P}}_+ = |+\rangle\langle +|$ and $\widehat{\mathbf{P}}_- = |-\rangle\langle -|$ as matrices, compute $P(+)$ and $P(-)$, and verify that they sum to one.

SOLUTION: The projectors are

$$\hat{\mathbf{P}}_+ = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \qquad \hat{\mathbf{P}}_- = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}.$$

The amplitudes are obtained from the inner products:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{14}}[(1+2i)+3] = \frac{4+2i}{\sqrt{28}} = \frac{2+i}{\sqrt{7}}, \qquad P(+) = \frac{4+1}{7} = \frac{5}{7}.$$

$$\langle -|\psi\rangle = \frac{1}{\sqrt{28}}[(1+2i)-3] = \frac{-2+2i}{\sqrt{28}} = \frac{-1+i}{\sqrt{7}}, \qquad P(-) = \frac{1+1}{7} = \frac{2}{7}.$$

$$P(+) + P(-) = \frac{5}{7} + \frac{2}{7} = 1 \checkmark$$

The same physical state gives a different probability distribution depending on which basis is measured. The state does not possess a value for both observables at once, and the measurement choice determines which question is asked.

d) *(for 7383 students only)* The expectation value of an observable can be obtained from the measurement probabilities. Using your results from parts (b) and (c), compute $\langle \mathbf{Z}\rangle = P(0) - P(1)$ and $\langle \mathbf{X}\rangle = P(+) - P(-)$. Then verify both by evaluating $\langle\psi|\mathbf{Z}|\psi\rangle$ and $\langle\psi|\mathbf{X}|\psi\rangle$ directly as matrix products.

SOLUTION: From the probabilities,

$$\langle \mathbf{Z}\rangle = \frac{5}{14} - \frac{9}{14} = -\frac{4}{14} = -\frac{2}{7}, \qquad \langle \mathbf{X}\rangle = \frac{5}{7} - \frac{2}{7} = \frac{3}{7}.$$

Evaluating directly, with $|\psi\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix}1+2i\\3\end{bmatrix}$ and $\langle\psi| = \frac{1}{\sqrt{14}}[1-2i \quad 3]$,

$$\langle\psi|\mathbf{Z}|\psi\rangle = \frac{1}{14}[1-2i \quad 3]\begin{bmatrix}1+2i\\-3\end{bmatrix} = \frac{(1+4)-9}{14} = \frac{-4}{14} = -\frac{2}{7}. \checkmark$$

$$\langle\psi|\mathbf{X}|\psi\rangle = \frac{1}{14}[1-2i \quad 3]\begin{bmatrix}3\\1+2i\end{bmatrix} = \frac{3(1-2i)+3(1+2i)}{14} = \frac{3-6i+3+6i}{14} = \frac{6}{14} = \frac{3}{7}. \checkmark$$

Both agree. The eigenvalues of $\mathbf{Z}$ and $\mathbf{X}$ are ± 1, so an expectation value is the probability-weighted average $(+1)P_{+1} + (-1)P_{-1}$, which is exactly the difference of the two probabilities.

4. **(Projective measurement of photon polarization — 18 pts for 5383; 17 pts for 7383)** The polarization of a single photon is a qubit. Take horizontal polarization as $|H\rangle = |0\rangle$ and vertical polarization as $|V\rangle = |1\rangle$. A linear polarizer whose transmission axis is at angle θ from the horizontal performs a projective measurement onto the state

$$|\theta\rangle = \cos\theta\,|H\rangle + \sin\theta\,|V\rangle.$$

A photon that passes through the polarizer emerges in the state $|\theta\rangle$, and a photon that is absorbed is simply lost.

a) Write the projector $\hat{\mathbf{P}}_\theta = |\theta\rangle\langle\theta|$ as a 2×2 matrix in terms of $\cos\theta$ and $\sin\theta$.

SOLUTION: The outer product of the column with its own row gives

$$\hat{\mathbf{P}}_\theta = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}[\cos\theta \quad \sin\theta] = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\\cos\theta\sin\theta & \sin^2\theta\end{bmatrix}.$$

All entries are real because $|\theta\rangle$ has real amplitudes, so no conjugation is needed when forming the bra.

b) A photon is prepared in the state $|H\rangle$ and is sent through a single polarizer set at $\theta = 60°$. Compute the probability that it is transmitted, and give the state of the photon after transmission. The general result you obtain here is known as Malus's law.

SOLUTION: The transmission probability is the squared magnitude of the projection amplitude:

$$\langle\theta|H\rangle = [\cos\theta \quad \sin\theta]\begin{bmatrix}1\\0\end{bmatrix} = \cos\theta, \qquad P(\text{transmitted}) = |\langle\theta|H\rangle|^2 = \cos^2\theta.$$

That relation is Malus's law. At $\theta = 60°$, where $\cos 60° = 1/2$,

$$P(\text{transmitted}) = \cos^2 60° = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25.$$

A transmitted photon emerges in the state $|60°\rangle = \frac{1}{2}|H\rangle + \frac{\sqrt{3}}{2}|V\rangle$. The polarizer does not filter photons that were already at 60°, it projects every transmitted photon into that state, which is what makes it a projective measurement rather than a sieve.

c) A photon prepared in $|H\rangle$ is sent toward a polarizer set at 90°. Compute the probability that it is transmitted. Now insert a second polarizer set at 45° *between* the source and the 90° polarizer, and compute the probability that the photon passes through both. Comment on the comparison, and explain why the result does not occur for a classical filter that only removes photons.

SOLUTION: With a single polarizer at 90°, the amplitude is $\langle 90°|H\rangle = \cos 90° = 0$, so

$$P(\text{transmitted}) = \cos^2 90° = 0.$$

No photon gets through, because $|H\rangle$ and $|V\rangle$ are orthogonal.

With the 45° polarizer inserted, the two stages are independent projective measurements and the probabilities multiply. The first stage takes $|H\rangle$ to $|45°\rangle$:

$$P_1 = \cos^2 45° = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}.$$

A photon that survives is now in the state $|45°\rangle$, not in $|H\rangle$. The second stage projects that onto $|90°\rangle$, and the angle between the two axes is again 45°:

$$P_2 = \cos^2(90° - 45°) = \cos^2 45° = \frac{1}{2}, \qquad P(\text{both}) = P_1 P_2 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Inserting an additional polarizer raises the transmission from 0 to 1/4. Adding an obstacle increases the throughput.

A classical filter that merely removes photons could never do this, since removing more photons can only reduce the count, and the probability of passing a sequence of filters would be non-increasing. The resolution is that the middle polarizer does not just remove photons, it changes the state of the ones it transmits. After the first stage the photon is genuinely in $|45°\rangle$, which has a nonzero overlap with $|90°\rangle$, whereas the original $|H\rangle$ had none. This is a direct experimental signature of state collapse under projective measurement.

d) *(for 7383 students only)* Show that $\widehat{\mathbf{P}}_\theta$ satisfies the three requirements of a projector associated with one outcome of a two-outcome measurement, namely that it is Hermitian, that $\widehat{\mathbf{P}}_\theta^2 = \widehat{\mathbf{P}}_\theta$, and that $\widehat{\mathbf{P}}_\theta + \widehat{\mathbf{P}}_{\theta+90^\circ} = \mathbf{I}$. State the physical meaning of the last identity.

SOLUTION: Write $c = \cos\theta$ and $s = \sin\theta$, so $\widehat{\mathbf{P}}_\theta = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$.

The matrix is real and symmetric, so its adjoint is itself and it is Hermitian.

For idempotence, multiply it out and factor using $c^2 + s^2 = 1$:

$$\widehat{\mathbf{P}}_\theta^2 = \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix} = \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \widehat{\mathbf{P}}_\theta . \checkmark$$

For completeness, use $\cos(\theta + 90^\circ) = -\sin\theta$ and $\sin(\theta + 90^\circ) = \cos\theta$, so $\widehat{\mathbf{P}}_{\theta+90^\circ} = \begin{bmatrix} s^2 & -sc \\ -sc & c^2 \end{bmatrix}$ and

$$\widehat{\mathbf{P}}_\theta + \widehat{\mathbf{P}}_{\theta+90^\circ} = \begin{bmatrix} c^2+s^2 & cs-sc \\ cs-sc & s^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} . \checkmark$$

Physically, the identity says that the two outcomes at θ and at $\theta + 90^\circ$ are exhaustive and mutually exclusive. Every photon is either transmitted by a polarizer at θ or by one at $\theta + 90^\circ$, never both and never neither, so the two probabilities always sum to one. A polarizing beam splitter realizes this directly by sending the two outcomes to two different output ports instead of absorbing one of them.

5. **(A simple Hamiltonian and its time evolution — 16 pts for 5383; 17 pts for 7383)** A qubit has the Hamiltonian operator

$$\hat{\mathbf{H}} = \hbar\omega|1\rangle\langle 1| = \hbar\omega\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},$$

where ω is a real positive constant. The state obeys the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$.

a) Verify that $\hat{\mathbf{H}}$ is Hermitian. Show that $|0\rangle$ and $|1\rangle$ are eigenvectors and give the corresponding energy eigenvalues. Then give the expectation value $\langle\hat{\mathbf{H}}\rangle$ for a general normalized state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

SOLUTION: The matrix is real and diagonal, so transposing and conjugating return the same matrix and $\hat{\mathbf{H}}^\dagger = \hat{\mathbf{H}}$. It is Hermitian, as any operator representing an observable must be.

$$\hat{\mathbf{H}}|0\rangle = \hbar\omega\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\cdot|0\rangle, \qquad \hat{\mathbf{H}}|1\rangle = \hbar\omega\begin{bmatrix}0\\1\end{bmatrix} = \hbar\omega|1\rangle.$$

So the energy eigenvalues are $E_0 = 0$ for $|0\rangle$ and $E_1 = \hbar\omega$ for $|1\rangle$. The two levels are separated by $\hbar\omega$.

For a general state,

$$\langle\hat{\mathbf{H}}\rangle = \langle\psi|\hat{\mathbf{H}}|\psi\rangle = \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix}0\\ \hbar\omega\beta\end{bmatrix} = \hbar\omega\,\beta^*\beta = \hbar\omega|\beta|^2.$$

This is the probability-weighted average of the two energies, $0\cdot|\alpha|^2 + \hbar\omega\cdot|\beta|^2$, as expected.

b) The time evolution operator is $\hat{\mathbf{U}}(t) = e^{-i\hat{\mathbf{H}}t/\hbar}$. Because $\hat{\mathbf{H}}$ is diagonal, the exponential may be taken entry by entry on the diagonal. Write $\hat{\mathbf{U}}(t)$ explicitly as a 2×2 matrix and verify that $\hat{\mathbf{U}}^\dagger\hat{\mathbf{U}} = \mathbf{I}$.

SOLUTION: For a diagonal matrix $\mathbf{D} = \mathrm{diag}(d_1, d_2)$ the power series gives $e^{\mathbf{D}} = \mathrm{diag}(e^{d_1}, e^{d_2})$, since every power of a diagonal matrix is diagonal with the entries raised to that power. Here

$$-\frac{i\hat{\mathbf{H}}t}{\hbar} = \begin{bmatrix}0 & 0\\ 0 & -i\omega t\end{bmatrix} \Rightarrow \hat{\mathbf{U}}(t) = \begin{bmatrix}e^0 & 0\\ 0 & e^{-i\omega t}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & e^{-i\omega t}\end{bmatrix}.$$

The adjoint is obtained by transposing and conjugating, which replaces $e^{-i\omega t}$ with $e^{+i\omega t}$:

$$\hat{\mathbf{U}}^\dagger\hat{\mathbf{U}} = \begin{bmatrix}1 & 0\\ 0 & e^{+i\omega t}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-i\omega t}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & e^0\end{bmatrix} = \mathbf{I}.\checkmark$$

c) The qubit is prepared in $|\psi(0)\rangle = |+\rangle$. Give $|\psi(t)\rangle$. Show that a computational-basis measurement gives $P(0) = P(1) = 1/2$ at every time t, but that a Hadamard-basis measurement gives $P(+) = \cos^2(\omega t/2)$ and $P(-) = \sin^2(\omega t/2)$.

SOLUTION: Applying the evolution operator,

$$|\psi(t)\rangle = \hat{\mathbf{U}}(t)|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & e^{-i\omega t}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{-i\omega t}\end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right).$$

In the computational basis each amplitude has modulus $1/\sqrt{2}$, and the phase factor has unit modulus, so

$$P(0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \qquad P(1) = \left|\frac{e^{-i\omega t}}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

for every t. In the Hadamard basis the two amplitudes interfere:

$$\langle +|\psi(t)\rangle = \frac{1}{\sqrt{2}}[1 \quad 1]\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{-i\omega t}\end{bmatrix} = \frac{1+e^{-i\omega t}}{2}.$$

$$P(+) = \left|\frac{1+e^{-i\omega t}}{2}\right|^2 = \frac{(1+\cos\omega t)^2 + \sin^2\omega t}{4} = \frac{2+2\cos\omega t}{4} = \frac{1+\cos\omega t}{2} = \cos^2\left(\frac{\omega t}{2}\right),$$

where the last step uses the half-angle identity $\cos^2(\theta/2) = (1+\cos\theta)/2$. The same computation with a minus sign gives

$$P(-) = \left|\frac{1-e^{-i\omega t}}{2}\right|^2 = \frac{1-\cos\omega t}{2} = \sin^2\left(\frac{\omega t}{2}\right), \qquad P(+) + P(-) = 1. \checkmark$$

The relative phase $e^{-i\omega t}$ is invisible to a computational-basis measurement but fully visible in the Hadamard basis. This is why the choice of measurement basis matters, and it is the mechanism behind interference-based quantum algorithms.

d) *(for 7383 students only)* Find the earliest time $t > 0$ at which a Hadamard-basis measurement is certain to give the outcome associated with $|-\rangle$. Then state what the energy difference $E_1 - E_0$ has to do with that time.

SOLUTION: Certainty requires $P(-) = \sin^2(\omega t/2) = 1$, so $\omega t/2 = \pi/2$ and

$$t = \frac{\pi}{\omega}.$$

At that instant $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\pi}|1\rangle\right) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$.

The energy difference sets the rate. Since $E_1 - E_0 = \hbar\omega$, the angular frequency of the relative phase is $\omega = (E_1 - E_0)/\hbar$, which is the Bohr frequency of the two-level system, and the time required is

$$t = \frac{\pi}{\omega} = \frac{\pi\hbar}{E_1 - E_0}.$$

A larger energy splitting drives the qubit from $|+\rangle$ to $|-\rangle$ faster. This is the reason gate speeds in a physical qubit are set by the energy-level separation of the device.

6. **(Introduction to classical information theory — 15 pts for 5383; 18 pts for 7383)** This problem is self-contained and uses only the definitions given here. Consider a discrete memoryless source that emits symbols from an alphabet $\{x_1, x_2, \ldots, x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$, where $\sum_j p_j = 1$. The **self-information** of the symbol x_j and the **Shannon entropy** of the source are defined as

$$I(x_j) = -\log_2 p_j \text{ bits}, \qquad H(X) = \sum_{j=1}^{n} p_j\, I(x_j) = -\sum_{j=1}^{n} p_j \log_2 p_j \text{ bits}.$$

The entropy is the average number of bits of information carried by one symbol, and a term with $p_j = 0$ contributes zero. You may use the values $\log_2 3 \approx 1.585$, $\log_2 5 \approx 2.322$, and $\log_2 7 \approx 2.807$.

a) A source emits four symbols $\{A, B, C, D\}$ with probabilities $\{1/2, 1/4, 1/8, 1/8\}$. Compute the self-information of each symbol in bits and compute the entropy $H(X)$ of the source. Compare $H(X)$ with the 2 bits per symbol that a fixed-length binary code would require, and state what the comparison means.

SOLUTION: Each probability is a negative power of two, so each self-information is an integer:

$$I(A) = -\log_2 \frac{1}{2} = 1, \qquad I(B) = -\log_2 \frac{1}{4} = 2, \qquad I(C) = I(D) = -\log_2 \frac{1}{8} = 3 \text{ bits}.$$

The rarer the symbol, the more information its occurrence conveys.

$$H(X) = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3) = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8} = \frac{7}{4} = 1.75 \text{ bits per symbol}.$$

A fixed-length code for four symbols needs $\log_2 4 = 2$ bits per symbol. The entropy is lower, by 0.25 bits per symbol, because the source is not uniform. The comparison means a variable-length code can represent this source using fewer bits on average than the fixed-length code, and that 1.75 bits per symbol is the floor below which no lossless code can go.

b) Consider a binary source with probabilities $\{p,\ 1-p\}$. Write $H(p)$ and evaluate it at $p = 0$, $p = 1/2$, and $p = 1$. State which value of p makes the entropy largest and which values make it zero, and explain both answers in terms of how uncertain the outcome is.

SOLUTION: The binary entropy function is

$$H(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

At $p = 0$ the first term is zero by the stated convention and the second is $-1 \cdot \log_2 1 = 0$, so $H(0) = 0$. The same argument gives $H(1) = 0$. At $p = 1/2$,

$$H\left(\frac{1}{2}\right) = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \text{ bit}.$$

The entropy is largest at $p = 1/2$ and is zero at $p = 0$ and $p = 1$. When $p = 0$ or $p = 1$ the outcome is known in advance, so observing it resolves no uncertainty and conveys no information. When $p = 1/2$ the two outcomes are equally likely, the observer has no basis for a better than even guess, and the uncertainty resolved by one observation is maximal at one full bit.

c) Return to the qubit state $|\psi\rangle$ of Problem 3. A computational-basis measurement produces outcomes 0 and 1 with the probabilities you found in Problem 3(b), and a Hadamard-basis measurement produces outcomes with the probabilities from Problem 3(c). Treating each measurement as a binary information source, compute the Shannon entropy of the outcome in each basis. Report both to three decimal places and comment on which measurement is less predictable.

SOLUTION: For the computational basis the probabilities are $5/14$ and $9/14$. Using $\log_2(5/14) = \log_2 5 - \log_2 14 \approx 2.322 - 3.807 = -1.485$ and $\log_2(9/14) = 2\log_2 3 - \log_2 14 \approx 3.170 - 3.807 = -0.637$,

$$H = -\frac{5}{14}(-1.485) - \frac{9}{14}(-0.637) = 0.531 + 0.410 = 0.940 \text{ bits.}$$

For the Hadamard basis the probabilities are $5/7$ and $2/7$. Using $\log_2(5/7) \approx 2.322 - 2.807 = -0.485$ and $\log_2(2/7) = 1 - 2.807 = -1.807$,

$$H = -\frac{5}{7}(-0.485) - \frac{2}{7}(-1.807) = 0.347 + 0.516 = 0.863 \text{ bits.}$$

The computational-basis measurement has the higher entropy, 0.940 bits against 0.863 bits, so it is the less predictable of the two. Its probabilities $5/14$ and $9/14$ sit closer to even odds than $5/7$ and $2/7$ do.

Note that one and the same physical state yields two different amounts of classical information depending on which measurement is performed. The number of bits extracted is a property of the measurement together with the state, not of the state alone.

d) *(for 7383 students only)* For the four-symbol source of part (a), construct a binary prefix code by assigning the codewords $A \to 0$, $B \to 10$, $C \to 110$, $D \to 111$. Verify that the code is a prefix code, compute its average codeword length in bits per symbol, and compare that length with $H(X)$. Then decode the received bit stream 1 0 0 1 1 1 1 1 0 and state why no separators between codewords are needed.

SOLUTION: A prefix code is one in which no codeword is a prefix of any other. Checking all pairs, 0 is not a prefix of 10, 110, or 111 because they all begin with 1; 10 is not a prefix of 110 or 111 because their second bits are 1 while the second bit of 10 is 0; and 110 and 111 differ in the third bit. The code is a prefix code.

The average codeword length is the probability-weighted average of the lengths:

$$\bar{L} = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3) = \frac{7}{4} = 1.75 \text{ bits per symbol.}$$

This equals $H(X) = 1.75$ bits exactly, so the code is optimal and no lossless code can do better for this source. Equality occurs here because every probability is a negative integer power of two, which lets each codeword length match the self-information of its symbol exactly.

Decoding 100111110 from the left, the decoder consumes bits until it recognizes a complete codeword and then starts over:

$$\underset{B}{10}\ \underset{A}{0}\ \underset{D}{111}\ \underset{C}{110} \Rightarrow \text{the message is } BADC.$$

No separators are required precisely because of the prefix property. As soon as the bits read so far form a codeword, that codeword cannot be the beginning of any longer one, so the decoder can commit to it immediately without ever needing to look ahead or backtrack. The code is said to be instantaneously decodable.