

Quantum Fourier Transform



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$$\mathbf{DFT}_3 = \frac{1}{(\sqrt{2})^3} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

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Classical Fourier Transform

- Transforms a continuous complex-valued function of one-variable to another continuous complex-valued function of another variable
- Commonly used in signal processing to transform functions of time to functions of frequency
- Expresses a function as a weighted linear combination of a set of periodic basis functions

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{+i\omega t} d\omega$$

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Discrete Fourier Transform

- The Fourier transform of a discrete function
- As an example, consider a polynomial function of the form:

$$p(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1}$$

- $p(x)$ can be represented as a column vector, $\mathbf{p}$

$$\mathbf{p} = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{n-1}]^T$$

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Polynomial Evaluation

- Suppose we want to evaluate the polynomial for a set of x -values

$$x_0, \quad x_1, \quad x_2, \quad \cdots, \quad x_{n-1}$$

- One way to find the set

$$p(x_0), \quad p(x_1), \quad p(x_2), \quad \cdots, \quad p(x_{n-1})$$

is to perform a matrix multiplication:

$$\begin{bmatrix} x_0^0 & x_0^1 & x_0^2 & \cdots & x_0^k & x_0^{k+1} & \cdots & x_0^{n-1} \\ x_1^0 & x_1^1 & x_1^2 & \cdots & x_1^k & x_1^{k+1} & \cdots & x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 & \cdots & x_2^k & x_2^{k+1} & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_k^0 & x_k^1 & x_k^2 & \cdots & x_k^k & x_k^{k+1} & \cdots & x_k^{n-1} \\ x_{k+1}^0 & x_{k+1}^1 & x_{k+1}^2 & \cdots & x_{k+1}^k & x_{k+1}^{k+1} & \cdots & x_{k+1}^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1}^0 & x_{n-1}^1 & x_{n-1}^2 & \cdots & x_{n-1}^k & x_{n-1}^{k+1} & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ p(x_2) \\ \vdots \\ p(x_k) \\ p(x_{k+1}) \\ \vdots \\ p(x_{n-1}) \end{bmatrix}$$

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Polynomial Evaluation

$$\begin{bmatrix} x_0^0 & x_0^1 & x_0^2 & \cdots & x_0^k & x_0^{k+1} & \cdots & x_0^{n-1} \\ x_1^0 & x_1^1 & x_1^2 & \cdots & x_1^k & x_1^{k+1} & \cdots & x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 & \cdots & x_2^k & x_2^{k+1} & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ x_k^0 & x_k^1 & x_k^2 & \cdots & x_k^k & x_k^{k+1} & \cdots & x_k^{n-1} \\ x_{k+1}^0 & x_{k+1}^1 & x_{k+1}^2 & \cdots & x_{k+1}^k & x_{k+1}^{k+1} & \cdots & x_{k+1}^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1}^0 & x_{n-1}^1 & x_{n-1}^2 & \cdots & x_{n-1}^k & x_{n-1}^{k+1} & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(x_0) \\ p(x_1) \\ p(x_2) \\ \vdots \\ p(x_k) \\ p(x_{k+1}) \\ \vdots \\ p(x_{n-1}) \end{bmatrix}$$

- Matrix contains rows that are a geometric series called a Vandermonde matrix Denoted as:

$$\mathbf{V}(x_0, x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_{n-1})$$

- For the DFT, the matrix coefficients are powers of the M^{th} roots of unity where the matrix is of size $M \times M$

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M^{th} Root of Unity

- For a fixed value of M , there are M different roots of unity:

$$e^{\frac{2\pi i k}{M}} = \omega^k \quad \omega = e^{\frac{2\pi i}{M}}$$

- The general form of the roots of unity is:

$$\sqrt[M]{1} = \omega^k, \quad k = 1, 2, \dots, M$$

- As an example, consider:

M	$\omega_k = \omega^{1/k}$	M	$\omega_k = \omega^{1/k}$
1	$e^{\frac{2\pi i 1}{1}}$	4	$e^{\frac{2\pi i 1}{4}}, e^{\frac{2\pi i 2}{4}}, e^{\frac{2\pi i 3}{4}}, e^{\frac{2\pi i 4}{4}}$
2	$e^{\frac{2\pi i 1}{2}}, e^{\frac{2\pi i 2}{2}}$	5	$e^{\frac{2\pi i 1}{5}}, e^{\frac{2\pi i 2}{5}}, e^{\frac{2\pi i 3}{5}}, e^{\frac{2\pi i 4}{5}}, e^{\frac{2\pi i 5}{5}}$
3	$e^{\frac{2\pi i 1}{3}}, e^{\frac{2\pi i 2}{3}}, e^{\frac{2\pi i 3}{3}}$	6	$e^{\frac{2\pi i 1}{6}}, e^{\frac{2\pi i 2}{6}}, e^{\frac{2\pi i 3}{6}}, e^{\frac{2\pi i 4}{6}}, e^{\frac{2\pi i 5}{6}}, e^{\frac{2\pi i 6}{6}}$

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Discrete Fourier Transform

- For $M=2^m$, using the Vandermonde matrix and the roots of unity:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & \omega^{1 \cdot 1} & \omega^{1 \cdot 2} & \dots & \omega^{1 \cdot j} & \dots & \omega^{1 \cdot (M-1)} \\ 1 & \omega^{2 \cdot 1} & \omega^{2 \cdot 2} & \dots & \omega^{2 \cdot j} & \dots & \omega^{2 \cdot (M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{k \cdot 1} & \omega^{k \cdot 2} & \dots & \omega^{k \cdot j} & \dots & \omega^{k \cdot (M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{(M-1) \cdot 1} & \omega^{(M-1) \cdot 2} & \dots & \omega^{(M-1) \cdot j} & \dots & \omega^{(M-1) \cdot (M-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p(\omega^0) \\ p(\omega^1) \\ p(\omega^2) \\ \vdots \\ p(\omega^k) \\ \vdots \\ p(\omega^{M-1}) \end{bmatrix}$$

- Matrix is denoted as **DFT**:

$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V(\omega^0, \omega^1, \omega^2, \dots, \omega^{M-1}) = \frac{1}{\sqrt{M}} [\omega^{jk}]$$

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DFT is Unitary

$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V(\omega^0, \omega^1, \omega^2, \dots, \omega^{M-1}) = \frac{1}{\sqrt{M}} [\omega^{jk}]$$

- The adjoint of **DFT** is:

$$\mathbf{DFT}^\dagger = \frac{1}{\sqrt{M}} [\omega^{kj}]^* = \frac{1}{\sqrt{M}} [\omega^{-jk}]$$

- To show unitary property, multiply:

$$[\mathbf{DFT}][\mathbf{DFT}^\dagger] = \frac{1}{M} \sum_{s=0}^{M-1} (\omega^{js} \omega^{-sk}) = \frac{1}{M} \sum_{s=0}^{M-1} \omega^{-s(k-j)}$$

- Diagonal Entry of **DFT** occurs when $j=k$:

$$[\mathbf{DFT}][\mathbf{DFT}^\dagger] = \frac{1}{M} \sum_{s=0}^{M-1} \omega^{-s(k-j)} = \frac{1}{M} \sum_{s=0}^{M-1} \omega^{-s(0)} = \frac{1}{M} \sum_{s=0}^{M-1} 1 = 1$$

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DFT is Unitary

$$\mathbf{DFT} = \frac{1}{\sqrt{M}} V(\omega^0, \omega^1, \omega^2, \dots, \omega^{M-1}) = \frac{1}{\sqrt{M}} [\omega^{jk}]$$

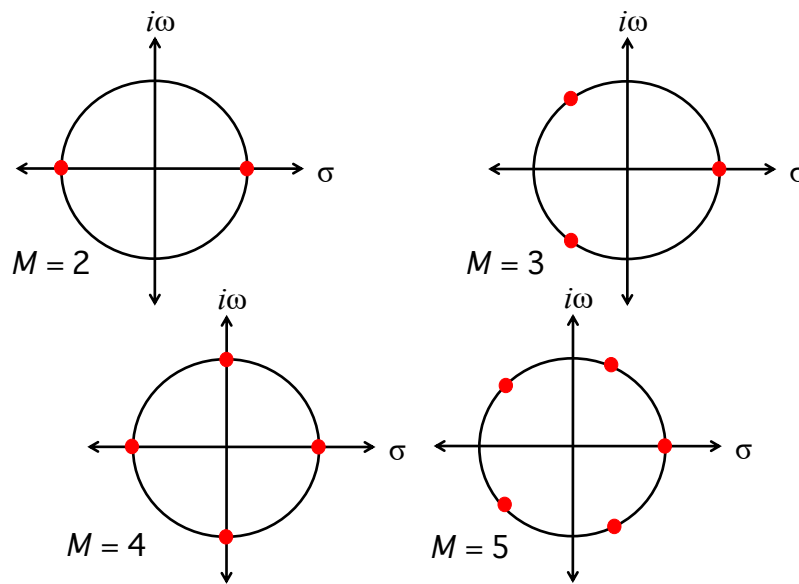
- Off-Diagonal Entry of **DFT** occurs when $j \neq k$
- Leads to a Geometric Progression that sums to Zero
- Therefore DFT is Unitary and:

$$[\mathbf{DFT}][\mathbf{DFT}^\dagger] = \mathbf{I}$$

- DFT matrix evaluates polynomials along equally spaced points on unit circle in complex plane
- Evaluations are periodic since the points go around the circle in a modular fashion

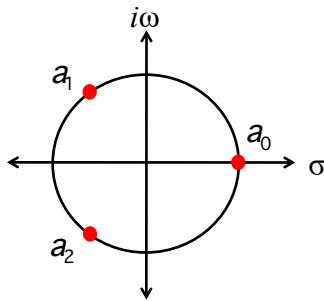
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Unit Circle on Complex Plane



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Three Roots of Unity



$$a_0 = e^{\frac{2\pi i 3}{3}} = \cos(2\pi) + i \sin(2\pi)$$

$$a_1 = e^{\frac{2\pi i 1}{3}} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

$$a_1 = e^{\frac{2\pi i 1}{3}} = \cos(120^\circ) + i \sin(120^\circ)$$

$$a_2 = e^{\frac{2\pi i 2}{3}} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

$$a_2 = e^{\frac{2\pi i 2}{3}} = \cos(240^\circ) + i \sin(240^\circ)$$

EXAMPLE:

$$(a_2)^3 = \left(e^{\frac{2\pi i 2}{3}}\right)^3 = e^{4\pi i} = \cos(720^\circ) + i \sin(720^\circ) = 1$$

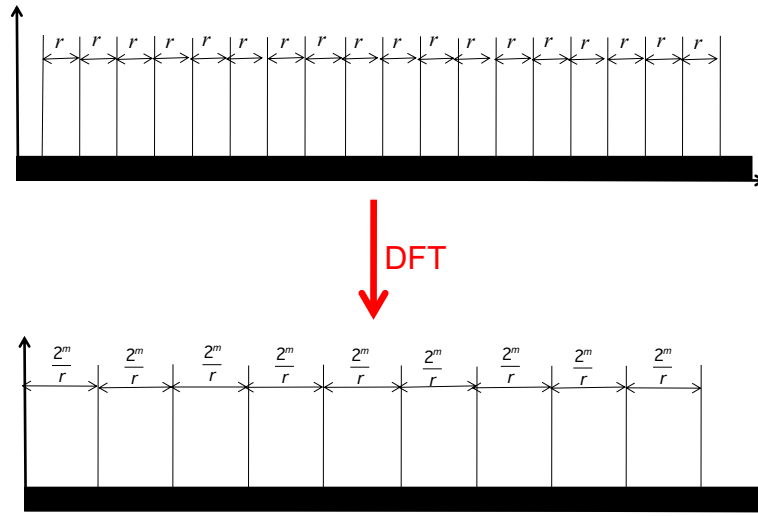
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DFT Properties

- Expresses a Function (polynomial) as a Function of (orthogonal) Roots of Unity
- Roots of Unity are Periodic
- If Function is Originally Periodic, DFT is Another Periodic Function with Modified Period
- If Original Function has Period r , Transformed Function has Period $2^m/r$
- Also Eliminates the Offset of the Original Periodic Function

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DFT Properties



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Quantum Fourier Transform (QFT)

- Variant of DFT Suitable for Implementation as a Quantum Circuit (Algorithm)
- Inverse of QFT is $\text{QFT}^\dagger$
- $\text{QFT}^\dagger$ is Crucial Component of Shor's Factoring Algorithm
- QFT/ $\text{QFT}^\dagger$ is Fast and Composed of "Small" Unitary Operators
- QFT/ $\text{QFT}^\dagger$ is Linear Operator that Transforms an Orthonormal Basis to Another

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QFT

- QFT is Linear Operator that Transforms an Orthonormal Basis

$$\{|0\rangle, |1\rangle, \dots, |j\rangle, \dots, |k\rangle, \dots, |N-1\rangle\}$$

- to the basis:

$$|j\rangle \mapsto \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{i2\pi jk}{M}} |k\rangle$$

- QFT Transforms a Quantum State into Another

$$|v\rangle \mapsto |w\rangle$$

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QFT (cont)

$$|v\rangle \mapsto |w\rangle$$

- where,

$$|v\rangle = \sum_{j=0}^{M-1} v_j |j\rangle \quad |w\rangle = \sum_{k=0}^{M-1} w_k |k\rangle$$

- probability amplitudes of transformed state w are the DFT of probability amplitudes of v :

$$w_k = \text{DFT}(v_j) = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} v_j e^{\frac{i2\pi jk}{M}}$$

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QFT (cont)

$$|v\rangle \mapsto |w\rangle$$

- when $M=2^n$, Consider the Binary Radix Polynomial Representations of j and k
 - (note subscripts intentionally written "backwards" here)

$$j = j_0 2^{n-1} + j_1 2^{n-2} + \dots + j_{n-2} 2^1 + j_{n-1} 2^0$$

$$k = k_0 2^{n-1} + k_1 2^{n-2} + \dots + k_{n-2} 2^1 + k_{n-1} 2^0$$

- QFT can be Rewritten as:

$$|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \dots \sum_{k_{n-1}=(0,1)} e^{i2\pi j \sum_{m=0}^{n-1} k_m 2^{-m}} |k_0 k_1 \dots k_m \dots k_{n-1}\rangle$$

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QFT (cont)

- Rearranging Following Result and Simplifying:

$$|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \dots \sum_{k_{n-1}=(0,1)} e^{i2\pi j \sum_{m=0}^{n-1} k_m 2^{-m}} |k_0 k_1 \dots k_m \dots k_{n-1}\rangle$$

- yields:

$$|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \dots \sum_{k_{n-1}=(0,1)} \bigotimes_{m=0}^{n-1} e^{i2\pi j k_m 2^{-m}} |k_m\rangle$$

$$|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ \sum_{k_m=(0,1)} e^{i2\pi j k_m 2^{-m}} |k_m\rangle \right\}$$

- k_m may only take values $\{0,1\}$:

$$|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \right\}$$

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QFT (cont)

- QFT: $|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \right\}$

- Simplifying:

$$|0\rangle + e^{i2\pi j 2^{-m}} |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i2\pi j 2^{-m}} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i2\pi j 2^{-m}} \end{bmatrix}$$

- Transformation on input $|j\rangle$ Rewritten as:

$$|j\rangle \mapsto \frac{1}{2^{n/2}} \begin{bmatrix} 1 \\ e^{i2\pi(j/2^0)} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^1)} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^2)} \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^{n-1})} \end{bmatrix}$$

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3-Qubit QFT Example

- QFT of Basis Vector $|000\rangle$ where $n=3$ and $j=0$:

$$|000\rangle \mapsto \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(\sqrt{2})^3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|000\rangle \mapsto \frac{1}{2^{3/2}} [|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle]$$

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3-Qubit QFT Example (cont)

- QFT of Basis Vector $|001\rangle$ where $n=3$ and $j=1$:

$$\begin{aligned}
 |001\rangle &\mapsto \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(1/2)} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi(1/4)} \end{bmatrix} \\
 &= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix} \\
 |001\rangle &\mapsto \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \\ 1 \\ -1 \\ -i \\ -i \end{bmatrix}
 \end{aligned}$$

$$|001\rangle \mapsto \frac{1}{2^{3/2}} [|000\rangle - |010\rangle + |100\rangle - |110\rangle + i(|001\rangle - |011\rangle + |101\rangle - |111\rangle)]$$

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QFT Quantum Cascade

- Recall: $|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \}$
- Can Use Regular Multiplication of ket Vectors

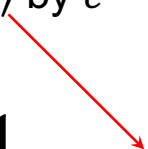
$$\begin{aligned}
 \bigotimes_{m=0}^{n-1} \{ |0\rangle + e^{i2\pi j 2^{-m}} |1\rangle \} &= \prod_{m=0}^{n-1} (|0\rangle + e^{i2\pi j 2^{-m}} |1\rangle) \\
 |j_0 j_1 \dots j_{n-1}\rangle &\mapsto \frac{1}{2^{n/2}} (|0\rangle + e^{i2\pi j 2^0} |1\rangle) (|0\rangle + e^{i2\pi j 2^1} |1\rangle) \dots (|0\rangle + e^{i2\pi j 2^{n-1}} |1\rangle)
 \end{aligned}$$

- Allows a Cascade to be Realized Using Hadamard Gates and Controlled-Rotation Gates
- Basis Vector Placed in Superposition using Hadamard First then Undergoes Rotations

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QFT Quantum Cascade (circuit)

- Recall General Phase Gate, R_m
- Transforms a Qubit by Multiplying its Projection on $|1\rangle$ by $e^{i2\pi/2^m}$

$$R_m = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^m} \end{bmatrix}$$


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QFT Quantum Cascade

- Distribute Scalar Factor
 $|j_0 j_1 \dots j_{n-1}\rangle \mapsto \frac{1}{2^{n/2}} (|0\rangle + e^{i2\pi j_0 2^0} |1\rangle) (|0\rangle + e^{i2\pi j_1 2^1} |1\rangle) \dots (|0\rangle + e^{i2\pi j_{n-1} 2^{n-1}} |1\rangle)$

- Becomes:

$$|j_0 j_1 \dots j_{n-1}\rangle = \left[\frac{1}{\sqrt{2}} (|0\rangle + e^{i2\pi j_0 2^0} |1\rangle) \right] \left[\frac{1}{\sqrt{2}} (|0\rangle + e^{i2\pi j_1 2^1} |1\rangle) \right] \dots \left[\frac{1}{\sqrt{2}} (|0\rangle + e^{i2\pi j_{n-1} 2^{n-1}} |1\rangle) \right]$$

- Consider the Single Qubit, $|j_m\rangle$:

$$|j_m\rangle = \left[\frac{1}{\sqrt{2}} (|0\rangle + e^{i2\pi j 2^m} |1\rangle) \right] = R_m \left[\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right]$$

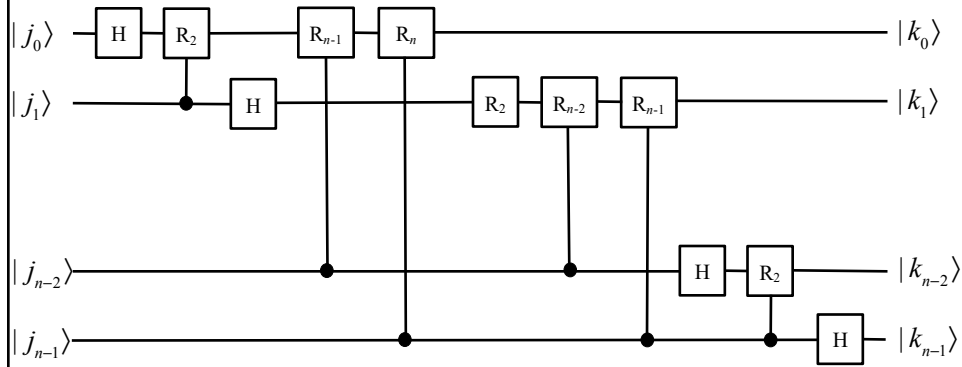
$$R_m = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^m} \end{bmatrix}$$

$$|j_m\rangle = R_m \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) |0\rangle$$

$$|j_m\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi j/2^m} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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QFT Quantum Cascade (circuit)



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3-Qubit QFT Example

- Recall the Identities: $e^{i\theta} = \cos \theta + i \sin \theta$

$$\cos\left(2k+1\right)\frac{\pi}{2}=0 \quad \sin\left(2k+1\right)\frac{\pi}{2}=1$$

$$e^{i\pi/2} = i \quad e^{i\pi/4} = \sqrt{i}$$

- Use the Following:

$$\omega = \sqrt{i} \quad \omega^2 = i \quad \omega^4 = -1$$

- Rotation Gates Become:

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$$

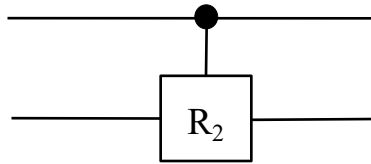
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Transfer Matrix – Controlled R^2

- Single Qubit R_2 Transfer Matrix:

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

- We want a Controlled Gate:



- The Transformations are:

$$\begin{aligned} |00\rangle &\mapsto |00\rangle & |10\rangle &\mapsto |10\rangle \\ |01\rangle &\mapsto |01\rangle & |11\rangle &\mapsto e^{i2\pi/2^2} |11\rangle \end{aligned}$$

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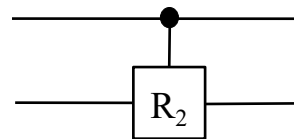
Transfer Matrix – Controlled R^2

- Controlled- R_2 Transfer Matrix using Projections:

$$\mathbf{G}_{R_2} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + e^{i2\pi/2^2} |11\rangle\langle 11|$$

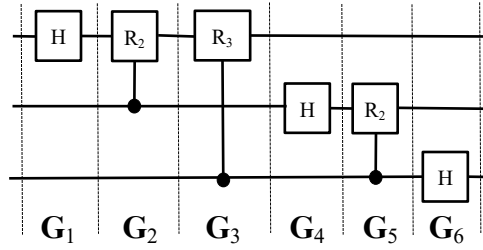
$$\mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i2\pi/2^2} \end{bmatrix}$$

$$\mathbf{G}_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i2\pi/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$



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QFT Quantum Cascade

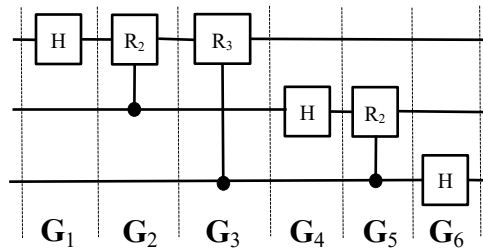


$$\mathbf{G}_1 = \mathbf{H} \otimes \mathbf{I} \otimes \mathbf{I} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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QFT Quantum Cascade (circuit)



$$\mathbf{G}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_4 & \mathbf{I}_4 \\ \mathbf{I}_4 & -\mathbf{I}_4 \end{bmatrix}, \quad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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QFT Quantum Cascade (circuit)

$$\mathbf{G}_2 = \mathbf{G}_{R_2} \otimes \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_4 & 0 \\ 0 & \mathbf{M}_{(2,4)} \end{bmatrix}$$

$$\mathbf{M}_{(2,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

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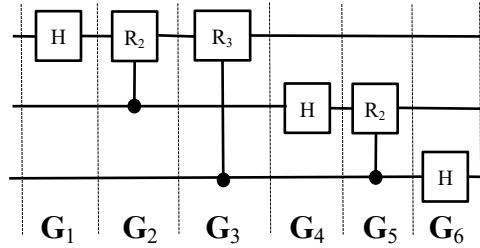
QFT Quantum Cascade (circuit)

$|000\rangle \mapsto |000\rangle, |001\rangle \mapsto |001\rangle, |010\rangle \mapsto |010\rangle, |011\rangle \mapsto |011\rangle$
 $|100\rangle \mapsto |100\rangle, |101\rangle \mapsto \omega |101\rangle, |110\rangle \mapsto |110\rangle, |111\rangle \mapsto \omega |111\rangle$

$$\mathbf{G}_3 = \begin{bmatrix} \mathbf{I}_4 & 0 \\ 0 & \mathbf{M}_{(3,4)} \end{bmatrix} \quad \mathbf{M}_{(3,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}$$

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QFT Quantum Cascade (circuit)

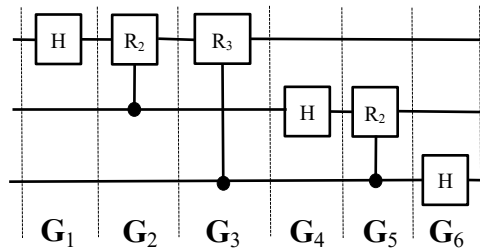


$$\mathbf{G}_4 = \mathbf{I} \otimes \mathbf{H} \otimes \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{G}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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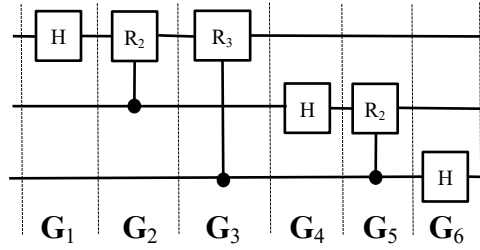
QFT Quantum Cascade (circuit)



$$\mathbf{G}_4 = \begin{bmatrix} \mathbf{M}_{(4,4)} & 0 \\ 0 & \mathbf{M}_{(4,4)} \end{bmatrix} \quad \mathbf{M}_{(4,4)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

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QFT Quantum Cascade (circuit)

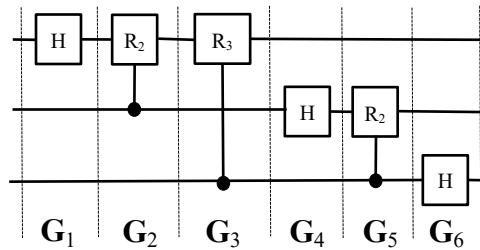


$$G_5 = I \otimes G_{R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{(5,4)} & 0 \\ 0 & \mathbf{M}_{(5,4)} \end{bmatrix}$$

$$\mathbf{M}_{(5,4)} = G_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}$$

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QFT Quantum Cascade (circuit)



$$G_6 = I \otimes I \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{M}_{(6,4)} & 0 \\ 0 & \mathbf{M}_{(6,4)} \end{bmatrix}$$

$$\mathbf{M}_{(6,4)} = G_{R_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

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3-Qubit QFT

- General Transformation is:

$$|w\rangle = G_6 G_5 G_4 G_3 G_2 G_1 |v\rangle$$

$$G_t = G_6 G_5 G_4 G_3 G_2 G_1$$

$$G_t = \frac{1}{(\sqrt{2})^3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \omega^2 & -1 & -\omega^2 & 1 & \omega^2 & -1 & -\omega^2 \\ 1 & -\omega^2 & -1 & \omega^2 & 1 & -\omega^2 & -1 & \omega^2 \\ 1 & \omega^1 & \omega^2 & \omega^3 & -1 & -\omega^1 & -\omega^2 & -\omega^3 \\ 1 & -\omega^1 & \omega^2 & -\omega^3 & -1 & \omega^1 & -\omega^2 & \omega^3 \\ 1 & \omega^3 & -\omega^2 & -\omega^5 & -1 & -\omega^3 & \omega^2 & \omega^5 \\ 1 & -\omega^3 & -\omega^2 & \omega^5 & -1 & \omega^3 & \omega^2 & -\omega^5 \end{bmatrix}$$

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3-Qubit QFT

- Note that: $-1 = e^{i2\pi/2} = \omega^4$

$$-\omega^2 = \omega^6 \quad -\omega^1 = \omega^5 \quad -\omega^3 = \omega^7$$

$$G_t = \frac{1}{(\sqrt{2})^3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

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3-Qubit QFT

- Consider the Permutation Matrix:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Application of Permutation Matrix to $\mathbf{G}_t$ Yields the $\mathbf{QFT}_3$ ($\mathbf{DFT}_3$) Matrix

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3-Qubit QFT

- DFT Matrix Results:

$$\mathbf{QFT}_3 = \mathbf{P}\mathbf{G}_t = \frac{1}{(\sqrt{2})^3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

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