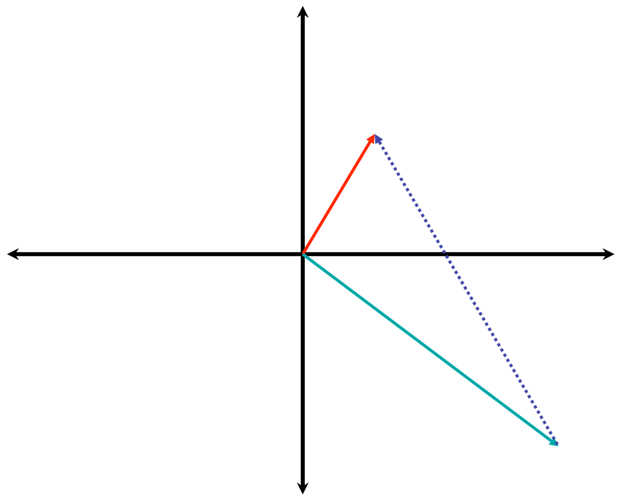
Linear Algebra



Vector Spaces and Operations

Group

- A Group is an Algebraic Structure Composed of a Set of elements with an Associated Binary Operator usually called Multiplication or the Group Product Operator

$$(G,*) \qquad\qquad *: G \times G \to G$$

- A Group Must Satisfy Three Conditions:

1. Associativity:

$$\forall (a,b,c) \in G \qquad a*(b*c) = (a*b)*c$$

2. Identity Element Exists:

$$\exists e \in G \qquad a*e = e*a = a \qquad \forall a \in G$$

3. Inverse Elements Exist:

$$\forall a \in G, \exists a^{-1} \in G \qquad a*a^{-1} = a^{-1}*a = e$$

Abelian Groups

- A Group that Also Obeys the Property of Commutativity is a Commutative or Abelian Group:

$$(G,*) \qquad *: G \times G \to G$$

4. Commutativity:

$$\forall (a,b) \in G \qquad a * b = b * a$$

- If Commutativity is not Obeyed, the Group is said to be non-Abelian or non-Commutative

Group Examples

- The Integers Under the Group Product Operation of Addition

$$(\mathbb{Z},+) \qquad \mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$$

  - Identity Element?
  - Inverse Elements?
  - Abelian?
- Positive Real Numbers Under Multiplication

$$(\mathbb{R},\bullet) \qquad \mathbb{R} = \{r \mid r > 0\}$$

  - Identity Element?
  - Inverse Elements?
  - Abelian?

Group Examples

- The Integers Under the Group Product Operation of Addition

$$(\mathbb{Z},+) \qquad \mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$$

  - Identity Element? 0
  - Inverse Elements? $\forall z_i \in \mathbb{Z}, z_i^{-1} = -z_i$; $z_i + -z_i = -z_i + z_i = 0$
  - Abelian? YES $z_i + z_j = z_j + z_i \quad \forall (z_i, z_j) \in \mathbb{Z}$

- Positive Real Numbers Under Multiplication

$$(\mathbb{R},\bullet) \qquad \mathbb{R} = \{r \mid r > 0\}$$

  - Identity Element? 1
  - Inverse Elements? $\forall r_i \in \mathbb{R}, r_i^{-1} = 1/r_i$; $r_i \bullet (1/r_i) = (1/r_i) \bullet r_i = 1$
  - Abelian? YES $r_i \bullet r_j = r_j \bullet r_i \quad \forall (r_i, r_j) \in \mathbb{R}$

More Group Examples

- The Set of Complex Numbers (excluding 0) under Multiplication are a Commutative Group
- Real/Complex Matrices under Matrix Multiplication are a Non-Abelian Group (matrix Multiplication is non-commutative)
- Rotation matrices (under multiplication) form a Group
  - in 2-D an Abelian Group
  - in higher dimensions non-Abelian Group
- The Symmetry Group: $\mathbb{S}_3$

Symmetry Group Example

- Consider Elements as Strings of Unique Objects
- Example: Group $(\mathbb{S}_3, \circ)$
- $\mathbb{S}_3$=(abc, bca, cab, bac, cba, acb)
- $\circ$ Represents the Permutation Operator
- 6 Objects in $\mathbb{S}_3$ Correspond to the Following Permutations

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \equiv 0 \quad \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \equiv 1 \quad \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \equiv 2$$

$$\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} \equiv 3 \quad \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \equiv 4 \quad \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \equiv 5$$

Symmetry Group Example

$\circ$ Represents the Permutation Operator

$$2 \circ 5 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \circ \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \circ \begin{pmatrix} b & a & c \\ b & c & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} = 3$$

$$5 \circ 2 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \circ \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \circ \begin{pmatrix} c & b & a \\ c & a & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} = 4$$

$$5 \circ 2 \neq 2 \circ 5$$

- Non-commutative - thus Non-Abelian

Symmetry Group Example

- Represents the Permutation Operator

$$3 \circ 4 = ??????$$

$$4 \circ 3 = ?????$$

Compute This on Paper

Symmetry Group Example

- Represents the Permutation Operator

$$3 \circ 4 = \begin{pmatrix} a\ b\ c \\ b\ c\ a \end{pmatrix} \circ \begin{pmatrix} a\ b\ c \\ c\ a\ b \end{pmatrix} = \begin{pmatrix} a\ b\ c \\ b\ c\ a \end{pmatrix} \circ \begin{pmatrix} b\ c\ a \\ a\ b\ c \end{pmatrix} = \begin{pmatrix} a\ b\ c \\ a\ b\ c \end{pmatrix} = 0$$

$$4 \circ 3 = \begin{pmatrix} a\ b\ c \\ c\ a\ b \end{pmatrix} \circ \begin{pmatrix} a\ b\ c \\ b\ c\ a \end{pmatrix} = \begin{pmatrix} a\ b\ c \\ c\ a\ b \end{pmatrix} \circ \begin{pmatrix} c\ a\ b \\ a\ b\ c \end{pmatrix} = \begin{pmatrix} a\ b\ c \\ a\ b\ c \end{pmatrix} = 0$$

$$3 \circ 4 = 4 \circ 3$$

- Still Non-commutative – in General

Field

- A Field F is set with two associated binary operators usually referred to as addition and multiplication
- A Field also Obeys the following Three Properties:
  1. Under Addition, F is an Abelian Group with Identity Element 0 Such That: $0 + a = a, \forall a \in F$
  2. Under Multiplication, the non-zero elements of F form an Abelian Group with Identity Element 1 Such That:
     $1 \bullet a = a, \forall a \in F \qquad 0 \bullet a = 0, \forall a \in F$
  3. Distributivity Holds:
     $$a \bullet (b + c) = a \bullet b + a \bullet c$$

Vector Space

- Vector Space Assumes the Existence of Three Objects:
  1. An Abelian Group $(V,+)$ whose Elements are Called Vectors and whose Product Operator is called Addition
  2. A Field F (usually $\mathbb{R}$ the real numbers, or $\mathbb{C}$ the complex numbers) whose Elements are called Scalars
  3. An Operation Called ***Multiplication with Scalars*** Denoted by • which associates to any scalar $c \in F$ and Vector $\alpha \in V$ another Vector $c \cdot \alpha \in V$ and has the following properties: $c \bullet (\alpha + \beta) = c \bullet \alpha + c \bullet \beta$

$$(c + c') \bullet \alpha = c \bullet \alpha + c' \bullet \alpha$$
$$(c \bullet c') \bullet \alpha = c \bullet (c' \bullet \alpha)$$
$$1 \bullet \alpha = \alpha$$

Linear Independence

- Given:

$$\{c_1, c_2, \ldots, c_n\} \in \mathbb{R} \qquad \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathbb{R}^n$$

- The set of n Vectors are ***Linearly Independent*** if:

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0 \Rightarrow c_i = 0 \forall i$$

No Solution for c_i Other Than all Equal 0

- Otherwise, the set of Vectors are Said to be ***Linearly Dependent***
- Linear Independence is a Property of a Specific Subset of Vectors

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- Is the Following set of Vectors Linearly Dependent?:

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

Compute This on Paper

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- Is the Following set of Vectors Linearly Dependent?: $\{\alpha_1, \alpha_2, \alpha_3\}$
- Check solution for: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- Check solution for: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$$0c_1 + 0c_2 + 1c_3 = 0$$
$$0c_1 + 2c_2 - 2c_3 = 0$$
$$1c_1 - 2c_2 + 1c_3 = 0$$

- Only Solution is: $c_1 = c_2 = c_3 = 0$
- Not Dependent (they are linearly Independent)

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- Are the Following set of Vectors Linearly Dependent?:

$$\{\alpha_2, \alpha_3, \alpha_4\}$$

Compute This on Paper

Linear Independence Example

$$0c_2 + 1c_3 + 4c_4 = 0 \qquad 2c_2 - 2c_3 + 2c_4 = 0 \qquad -2c_2 + 1c_3 + 3c_4 = 0$$

$$c_3 = -4c_4$$

$$2c_2 - 2(-4c_4) + 2c_4 = 0$$

$$c_2 = -5c_4$$

$$-2(-5c_4) - 4c_4 + 3c_4 = 0$$

$$c_4 = 0$$

$$c_2 = c_3 = c_4 = 0$$

- Not Dependent (they are linearly Independent)

Linear Independence Example

$$\alpha_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

- The following sets are Linearly Independent:

$$\{\alpha_1, \alpha_2, \alpha_3\} \qquad \alpha_4 = 9\alpha_1 + 5\alpha_2 + 4\alpha_3$$

$$\{\alpha_2, \alpha_3, \alpha_4\} \quad \alpha_1 = \left(-\frac{5}{9}\right)\alpha_2 + \left(-\frac{4}{9}\right)\alpha_3 + \left(\frac{1}{9}\right)\alpha_4$$

- The following set is Linearly Dependent:

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

Real Vector Spaces

- Consider an n-dimensional Vector Space:
- If, for all pairs of vectors α and β, an associated real number exists (α, β) such that the Following conditions are satisfied:

$$(\alpha, \beta) = (\beta, \alpha)$$

$$(c\alpha, \beta) = c(\alpha, \beta) \text{ if } c \in \mathbb{R}$$

$$(\alpha + \gamma, \beta) = (\alpha, \beta) + (\gamma, \beta) \quad \forall \gamma \in \mathbb{R}^n$$

$$(\alpha, \alpha) \geq 0 \text{ such that } \quad (\alpha, \alpha) = 0 \text{ if and only if } \alpha = 0$$

- Then, we have an n-dimensional ***Euclidean*** Vector Space
- (α, β) is the ***Inner Product*** of Vectors α and β

Euclidean Vectors

- Length of a Euclidean Vector:

$$|\alpha| = \sqrt{(\alpha,\alpha)}$$

- Angle between the two vectors α and β:

$$\theta = \cos^{-1}\frac{(\alpha,\beta)}{|\alpha||\beta|} \qquad \cos(\theta) = \frac{(\alpha,\beta)}{|\alpha||\beta|}$$

- If $(\alpha, \beta)=0$, then α and β are orthogonal and:

$$\theta = \pi / 2 = 90^{\circ}$$

Orthogonal Basis Sets

- Consider a set of n Vectors:

$$\mathfrak{E} = \{e_1, e_2, \ldots, e_n\}$$

- This set forms an ***Orthogonal Basis*** of the n-Dimensional Vector Space if:

$$(e_i, e_j) = 0, \forall i \neq j$$

- This set forms an ***Orthonormal Basis*** of the n-Dimensional Vector Space if:

$$(e_i, e_j) = \delta_{i,j} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Euclidean Space Basis

- All Vectors in a Euclidean Space may be Represented as a Linear Combination of the Orthonormal Basis Vectors:

$$\alpha = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$$

$$\beta = b_1 e_1 + b_2 e_2 + \ldots + b_n e_n$$

- Since: $(e_i, e_j) = \delta_{i,j}$
- Then: $(\alpha, e_i) = a_i$
- Thus: $(\alpha, \beta) = \sum_{i=1}^{n} a_i b_i$

This form of the inner product sometimes called the dot product

Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

- $\mathbf{A}$ maps Vectors from Vector Space of Dimension n to Vector Space of Dimension m
- When $\mathbf{A}$ is a Square Matrix it Represents a Linear Mapping to Itself
- Each Row of $\mathbf{A}$ is a Row Vector and Each Column is a Column Vector
- Row/Column Vectors Span the Domain/Range Vector Spaces

Elementary Row Operations

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

1. Any row may be interchanged with any other
2. Any row may be replaced by itself multiplied by a constant
3. Any row may be replaced by the column-wise sum of itself and a multiple of another row

Two Matrices are Row-Equivalent if one is Obtained from the Other by a Finite Sequence of Row Operations

Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- Identity Matrix is $n \times n$ Square Matrix whose Row-Vectors and Column Vectors Form an Orthonormal Basis for the n-dimensional Euclidean Vector Space
- Permutation Matrix is an Identity Matrix that has Undergone an Arbitrary Series of Row Interchanges

Matrix Determinant

- Determinant of a Matrix is Denoted as:

$$|\mathbf{A}| \qquad \det(\mathbf{A})$$

- Examples of Determinant Computation:

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|\mathbf{A}_1| = a_{11} \qquad |\mathbf{A}_2| = a_{11}a_{22} - a_{12}a_{21}$$

$$|\mathbf{A}_3| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrix Operations

- Transpose of a matrix, reflection about the diagonal:

$$\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{ji} \end{bmatrix}$$

- Determinant of $\mathbf{A}$ is Equal to Determinant of $\mathbf{A}^{\mathrm{T}}$
- If Two or More Rows (or columns) of $\mathbf{A}$ are Equivalent then $|\mathbf{A}|=0$
- A Square $n \times n$ Matrix is Triangular When:

$$\forall i > j, a_{ij} = 0 \text{ (upper triangular)}$$
$$\forall i < j, a_{ij} = 0 \text{ (lower triangular)}$$

- Determinant of Triangular Matrix $\mathbf{A}_{\mathrm{tri}}$

$$\det(\mathbf{A}_{\mathrm{tri}}) = |\mathbf{A}_{\mathrm{tri}}| = a_{11} \bullet a_{22} \bullet \ldots \bullet a_{nn}$$

Rank of a Matrix

- Rank of a Matrix is an Integer that is Equal to Number of Linearly Independent Row (Column) Vectors of a Square Matrix
- All Full Rank Matrices may be Converted into Triangular Matrices through Elementary Row Operations
- A Full Rank Matrix Must have a non-zero Determinant
- A non-Square Matrix Cannot Have a Rank Larger than $\min(m,n)$

Characteristic Equation

- Characteristic Equation of a Matrix $\mathbf{A}$ is:

$$c(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}|$$

- Roots of the Characteristic Equation yield the characteristic values, or eigenvalues of $\mathbf{A}$:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

- Eigenvalues of $\mathbf{A}$ are Scalar Multiples of Eigenvectors of $\mathbf{A}$:
- Eigenvectors of $\mathbf{A}$ are Those Vectors, when Transformed by $\mathbf{A}$ are Equivalent to Themselves by a Scale Factor λ

Trace of a Matrix

- Trace of a Matrix $\mathbf{A}$ is:

$$\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$

- Given two matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$$

$$\mathrm{Tr}(\mathbf{A}+\mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B})$$

$$\mathrm{Tr}(c\mathbf{A}) = c\,\mathrm{Tr}(\mathbf{A})$$

$$\mathrm{Tr}(\mathbf{SAS}^{\dagger}) = \mathrm{Tr}(\mathbf{S}^{\dagger}\mathbf{SA}) = \mathrm{Tr}(\mathbf{A})$$

Similarity Transform

Why all this vector stuff?

- Vectors used to Describe the State of a Quantum System
- A Quantum System is a Collection of Qubits
- Quantum Systems Evolve over Time
- Evolution means the quantum state of the qubits change
- Evolution can be Modeled with Transformation Matrices
- Quantum State vectors Exist in the Complex Vector Space

Complex Numbers

- Complex Numbers have a REAL and IMAGINARY Component and Exist in the Complex Field $\mathbb{C}$

$$c \in \mathbb{C} \qquad c = a + ib \qquad i^2 = -1$$

$$\mathrm{Re}(c) = a \in \mathbb{R} \qquad \mathrm{Im}(c) = b \in \mathbb{R}$$

- Recall Euler's Identity: $Ke^{i\theta} = K\cos\theta + iK\sin\theta$
- Phasor Notation: $K\angle\theta \quad a = K\cos\theta \quad b = K\sin\theta$
- Complex Conjugate: $c^* \in \mathbb{C}$

$$c = a \pm ib \Rightarrow c^* = a \mp ib$$

- Note:

$$c \bullet c^* = c^* \bullet c = a^2 + b^2$$

$$|c| = \sqrt{c \bullet c^*} = \sqrt{c^* \bullet c} = \sqrt{a^2 + b^2}$$

Inner Products in Complex Fields

- Satisfy Three Conditions:

$$(\alpha,\beta) = (\beta,\alpha)^*$$

$$(\alpha,\alpha) \geq 0$$

$$(\alpha,\alpha) = 0 \Rightarrow \alpha = 0$$

- Inner Products Induce Concept of a ***Norm***

$$\|\alpha\|$$

- Norm is a Measure of Vector Length or Magnitude
- Previous Example with Inner Product Defined the Euclidean Norm
- Norms can Exist when Inner Products do Not
- Finite Dimensional Vector Spaces with Norms are Banach Spaces

Vector Norms in Complex Fields

- Satisfy Three Conditions:

$$\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$$
$$\|c \bullet \alpha\| = |c|\,\|\alpha\|$$
$$\|\alpha\| = 0 \Rightarrow \alpha = 0$$

- Other types of Norms:
  - Manhattan Norm $\|\alpha\| = \sum_{i=1}^{n} |a_i|$
  - p-Norm $\|\alpha\|_p = \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}}$
  - Infinity-Norm $\|\alpha\|_\infty = \max(|a_1|,\ldots,|a_n|)$

What happens when p=1,2?

Adjoint Operator

- Denoted by Superscript "dagger" Symbol †
- Applicable to Vectors and Matrices
  - Vector $\alpha^\dagger = (\alpha^*)^{\mathrm{T}}$
  - Matrix $\mathbf{A}^\dagger = (\mathbf{A}^*)^{\mathrm{T}}$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}$$

Find: $\mathbf{A}^\dagger = ?$

Compute This on Paper

Adjoint Operator

- Denoted by Superscript "dagger" Symbol †
- Applicable to Vectors and Matrices
  - Vector $\alpha^{\dagger} = (\alpha^*)^{\mathrm{T}}$
  - Matrix $\mathbf{A}^{\dagger} = (\mathbf{A}^*)^{\mathrm{T}}$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}$$

$$\mathbf{A}^{\dagger} = \begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}^{\dagger} = \left(\begin{bmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{bmatrix}^{*} \right)^{\mathrm{T}}$$

$$\mathbf{A}^{\dagger} = \begin{bmatrix} 1+5i & 1-i \\ 1-3i & -7i \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1+5i & 1-3i \\ 1-i & -7i \end{bmatrix}$$

Adjoint Properties

- Adjoints of Identity Matrices are Themselves
- Adjoints of Real-Valued Matrices are Equivalent to the Transpose
- An Operator Defined by a Transformation Matrix $\mathbf{A}$ is ***Normal*** if $\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}$
- A Matrix $\mathbf{A}$ is said to be ***Hermitian*** is it is ***Self-adjoint*** Meaning:

$$\mathbf{A} = \left[a_{ij} \right] \qquad \mathbf{A}^{\dagger} = \left[a_{ji}^{*} \right]$$

$$a_{ij} = a_{ji}^{*}$$

Unitary Matrices

- A Square Matrix is Unitary if:

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_n$$

- Unitary Matrix Properties:

$$(\mathbf{U}\alpha, \mathbf{U}\beta) = (\alpha, \beta)$$

$$\mathbf{U}^{-1} = \mathbf{U}^\dagger$$

$$\mathrm{Rank}(\mathbf{U}) = n$$

Row (Column) Vectors Form an Orthonormal Basis for $\mathbb{C}^n$

$$|\lambda_i| = 1$$

$$|\det(\mathbf{U})| = 1$$

Complex Vector Spaces

- Hilbert Space is infinite-dimensional vector space with inner product and associated norm
- Quantum Computing Literature Traditionally Refers to n-dimensional Complex Euclidean Vector Space as a Hilbert Space (technically correct)
- FOR OUR PURPOSES: Hilbert Space: n-dimensional vector space over the field of complex numbers with an inner product and associated norm

Dirac Notation

- Traditional Notation for Representing Vectors in Quantum Mechanics is due to Paul Dirac
- Basis Vectors for n-Dimensional Hilbert Vector Space $\mathbb{H}^n$ as kets and bras:

$$\{|0\rangle, |1\rangle, \ldots, |i\rangle, \ldots, |n-1\rangle\}$$

$$\{\langle 0|, \langle 1|, \ldots, \langle i|, \ldots, \langle n-1|\}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, |i\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, |n\text{-}1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dirac Notation

- Basis vectors as Bras are Row Vectors that Span $\mathbb{H}^n$

$$\langle 0| = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix}$$

$$\langle 1| = \begin{bmatrix} 0 & 1 & \ldots & 0 & \ldots & 0 \end{bmatrix}$$

$$\vdots$$

$$\langle i| = \begin{bmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 \end{bmatrix}$$

$$\vdots$$

$$\langle n-1| = \begin{bmatrix} 0 & 0 & \ldots & 0 & \ldots & 1 \end{bmatrix}$$

Dirac Notation of Vectors

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \ldots + \alpha_i|i\rangle + \ldots + \alpha_{n-1}|n\text{-}1\rangle$$

complex values

Each ket Vector has a Dual bra Vector related by Hermitian Conjugation

$$|\psi\rangle = (\langle\psi|)^\dagger \qquad \langle\psi| = (|\psi\rangle)^\dagger$$

$$\langle\psi| = \alpha_0^*\langle 0| + \alpha_1^*\langle 1| + \ldots + \alpha_i^*\langle i| + \ldots + \alpha_{n-1}^*\langle n\text{-}1|$$

Dirac Notation of Vectors

$$|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$

$$|\psi\rangle = (\langle\psi|)^\dagger \qquad \langle\psi| = (|\psi\rangle)^\dagger$$

$$\langle\psi| = \begin{bmatrix} \alpha_0^* & \alpha_1^* & \ldots & \alpha_i^* & \ldots & \alpha_{n-1}^* \end{bmatrix}$$

Inner Product in Hilbert Space

- Inner Product of Two Vectors: $|\psi_a\rangle, |\psi_b\rangle \in \mathbb{H}^n$
- Denoted as: $\langle\psi_a|\psi_b\rangle$
- Properties:

1. Inner Product with Same Vector: $\langle\psi|\psi\rangle \in \mathbb{R}^n$
2. Linearity $|\psi_a\rangle, |\psi_b\rangle, |\psi_c\rangle \in \mathbb{H}^n \qquad a,b,c \in \mathbb{C}$

$$\langle\psi_a|(c|\psi_b\rangle) = c\langle\psi_a|\psi_b\rangle$$

$$(a\langle\psi_a| + b\langle\psi_b|)|\psi_c\rangle = a\langle\psi_a|\psi_c\rangle + b\langle\psi_b|\psi_c\rangle$$

$$\langle\psi_c|(a|\psi_a\rangle + b|\psi_b\rangle) = a\langle\psi_c|\psi_a\rangle + b\langle\psi_c|\psi_b\rangle$$

3. Skew Symmetry

$$\langle\psi_a|\psi_b\rangle = \langle\psi_b|\psi_a\rangle^*$$

Inner Product Example

- Inner Product of Two Vectors: $|\psi_a\rangle, |\psi_b\rangle \in \mathbb{H}^4$

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$$

$$|\psi_b\rangle = \beta_0|0\rangle + \beta_1|1\rangle + \beta_2|2\rangle + \beta_3|3\rangle$$

$$\langle\psi_a|\psi_b\rangle = \begin{bmatrix} \alpha_0^* & \alpha_1^* & \alpha_2^* & \alpha_3^* \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \alpha_0^*\beta_0 + \alpha_1^*\beta_1 + \alpha_2^*\beta_2 + \alpha_3^*\beta_3$$

Inner Product Example

- Inner Product of Two Vectors: $|\psi_a\rangle, |\psi_b\rangle \in \mathbb{H}^2$

$$|\psi_a\rangle = (1+i)|0\rangle + (2-3i)|1\rangle$$

$$|\psi_b\rangle = (1-2i)|0\rangle + (3+2i)|1\rangle$$

Compute this on paper

Inner Product Example

- Inner Product of Two Vectors: $|\psi_a\rangle, |\psi_b\rangle \in \mathbb{H}^2$

$$|\psi_a\rangle = (1+i)|0\rangle + (2-3i)|1\rangle$$

$$|\psi_b\rangle = (1-2i)|0\rangle + (3+2i)|1\rangle$$

$$\langle\psi_a|\psi_b\rangle = (1+i)*(1-2i) + (2-3i)*(3+2i)$$

$$\langle\psi_a|\psi_b\rangle = (1-i)(1-2i) + (2+3i)(3+2i)$$

$$\langle\psi_a|\psi_b\rangle = (1-3i-2) + (6+13i-6)$$

$$\langle\psi_a|\psi_b\rangle = (-1-3i) + (0+13i)$$

$$\langle\psi_a|\psi_b\rangle = -1+10i$$

$$\langle\psi_b|\psi_a\rangle = \langle\psi_a|\psi_b\rangle* = -1-10i$$

Inner Product Example

- Inner Product of Vector with itself: $|\psi_a\rangle \in \mathbb{H}^4$

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$$

$$\langle\psi_a|\psi_a\rangle = \begin{bmatrix}\alpha_0^* & \alpha_1^* & \alpha_2^* & \alpha_3^*\end{bmatrix}\begin{bmatrix}\alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3\end{bmatrix} = \alpha_0^*\alpha_0 + \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 + \alpha_3^*\alpha_3$$

$$\langle\psi_a|\psi_a\rangle = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2$$

Inner Product Example

- Inner Product of Vector with itself: $|\psi_a\rangle \in \mathbb{H}^2$

$$|\psi_a\rangle = (1+2i)|0\rangle + (4-3i)|1\rangle$$

$$\langle\psi_a|\psi_a\rangle = (1+2i)^*(1+2i) + (4-3i)^*(4-3i)$$

$$\langle\psi_a|\psi_a\rangle = (1-2i)(1+2i) + (4+3i)(4-3i)$$

$$\langle\psi_a|\psi_a\rangle = (1+4) + (16+9) = 5 + 25 = 30$$

$$\alpha_0 = (1+2i) \qquad \alpha_1 = (4-3i)$$

$$|\alpha_0| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad |\alpha_1| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\langle\psi_a|\psi_a\rangle = |\alpha_0|^2 + |\alpha_1|^2 = \left(\sqrt{5}\right)^2 + (5)^2 = 30$$

Inner Product Example

- Orthogonality:

$$|\psi_a\rangle \perp |\psi_b\rangle \Rightarrow \langle \psi_a | \psi_b \rangle = 0$$

$$|\psi_a\rangle \perp |\psi_b\rangle \Rightarrow |\psi_b\rangle \perp |\psi_a\rangle$$

- Normal ***Unitary*** Basis of n-dimensional basis:

$$\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_i\rangle, \ldots, |\psi_n\rangle\}$$

$$\| |\psi_i\rangle \| = 1$$

$$\langle \psi_i | \psi_j \rangle = 0 \forall i \neq j$$