Tensor and Outer Products

$$|\psi_a\rangle\langle\psi_b| = \begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_i\\ \vdots\\ \alpha_n\end{bmatrix} \otimes \begin{bmatrix}\beta_0 & \beta_1 & \cdots & \beta_i & \cdots & \beta_n\end{bmatrix}$$

Topics in Matrix and Tensor Algebra

Vector Tensor Product

- Consider the Following Two Vectors:

$$\alpha = \begin{bmatrix}a_1\\ a_2\end{bmatrix} \qquad \beta = \begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$\alpha \otimes \beta = \begin{bmatrix}a_1b_1\\ a_1b_2\\ a_2b_1\\ a_2b_2\end{bmatrix}$$

Vector Tensor Product

- Consider the Basis Vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|1\rangle|0\rangle|1\rangle = |101\rangle = |1\rangle \otimes |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |5\rangle$$

Matrix Tensor Product

- Origin in Group Theory - Important Applications in Quantum Mechanics
- Consider the Following Two Matrices:

$$\mathbf{A} = \left[a_{ij} \right] \text{ of order } (m \times n) \qquad \mathbf{B} = \left[b_{ij} \right] \text{ of order } (r \times s)$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} \text{ is of order } (mr \times ns)$$

Tensor Product Example

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

Tensor Product Properties

$$\mathbf{A} \otimes \mathbf{B} = \left[a_{ij}\mathbf{B} \right]$$

$$\mathbf{A} \otimes (\alpha\mathbf{B}) = \alpha(\mathbf{A} \otimes \mathbf{B})$$

$$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$$

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$$

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

$$\left|\mathbf{A} \otimes \mathbf{B}\right| = \left|\mathbf{A}\right|^{m} \left|\mathbf{B}\right|^{n} \quad \text{for } \mathbf{A}\ (n \times n) \text{ and } \mathbf{B}\ (m \times m)$$

$$\mathbf{A} \otimes \mathbf{B} = \mathbf{U}_1 (\mathbf{B} \otimes \mathbf{A}) \mathbf{U}_2 \quad \text{where } \mathbf{U}_1, \mathbf{U}_2 \text{ are permutation matrices}$$

$$\mathrm{Tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{B})$$

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_{\mathbf{m}} + \mathbf{I}_{\mathbf{n}} \otimes \mathbf{B} \text{ where } \mathbf{A} \text{ is } (n \times n) \text{ and } \mathbf{B} \text{ is } (m \times m)$$

Kronecker Sum

Outer Product

- Special Case of the Tensor Product
- Product is $m \times n$ Matrix Resulting from $m \times 1$ and $1 \times n$

$$|\psi_a\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \qquad |\psi_b\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$|\psi_a\rangle\langle\psi_b| = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1^* & \beta_2^* \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1^* & \alpha_1\beta_2^* \\ \alpha_2\beta_1^* & \alpha_2\beta_2^* \end{bmatrix}$$

Quantum State

Complete Description of a Quantum System

- Quantum State Represented by a Vector
- Quantum State Vector has a Norm of 1 in the Hilbert Space
- Traditional Notation for Quantum State:

$$|\psi_a\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle ... + \alpha_i\,|i\rangle ... + \alpha_{n-1}\,|n-1\rangle$$

Quantum State Properties

- Two States are Equivalent if:

$$|\psi_a\rangle = c\,|\psi_{a'}\rangle$$

- Where:

$$c \in \mathbb{C} \qquad |c| = 1$$

- Norm (length) of State Vector:

$$\sqrt{\langle\psi_a|\psi_a\rangle}$$

- Because State Vectors are Normalized:

$$\sqrt{\langle\psi_a|\psi_a\rangle} = \langle\psi_a|\psi_a\rangle = \sum_{i=0}^{n-1}|\alpha_i|^2 = 1$$

Quantum State Properties

- State Vectors are Normalized, thus Direction not Length Define State
- Quantum State is Really a ***ray*** in Hilbert Space
- Ray is an Element of Direction Only
- Traditional to Utilize Normalized State Vectors to Represent State

$$\langle\psi_a|\psi_a\rangle = 1$$

Quantum State Properties

- Consider the Following ***Phase Factor***:

$$e^{i\gamma} = \cos\gamma + i\sin\gamma$$
$$|e^{i\gamma}| = \sqrt{\cos^2\gamma + \sin^2\gamma} = 1$$

- Consider the Following ***Quantum State Vectors***: $|\psi_a\rangle$ $e^{i\gamma}|\psi_a\rangle$
- These Vectors Describe the Same Quantum State
- γ Represents the Relative Phase

Inner Products of State Vectors

- Inner Product Represents ***Generalized Angle*** Between States:

$$\langle\psi_a|\psi_b\rangle$$

  - Orthogonal States:

$$\langle\psi_a|\psi_b\rangle = 0$$

  - Equivalent States:

$$\langle\psi_a|\psi_b\rangle = 1$$

- Inner Product is a Complex Number
- Measure of ***Relative Orthogonality***:

$$|\langle\psi_a|\psi_b\rangle|$$

State Vector Bases

- Can Represent Quantum State Vector as Linear Combination of Unit Vectors:

$$\{|0\rangle, |1\rangle, \ldots, |i\rangle, \ldots, |n-1\rangle\}$$

- EXAMPLE: $\mathbb{H}^2$

$$|\psi_a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$\langle\psi_a | = \alpha_0^* \langle 0 | + \alpha_1^* \langle 1 |$$

Alternative Bases

- EXAMPLE: H^2 $\quad |\psi_a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

$$|\psi_a\rangle = \sigma_x |x\rangle + \sigma_y |y\rangle$$

$$|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Find σ_x, σ_y in Terms of α_0, α_1

Compute this on paper

Alternative Bases

- EXAMPLE: $\mathbb{H}^2$ $\quad |\psi_a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

$$|\psi_a\rangle = \sigma_x |x\rangle + \sigma_y |y\rangle$$

Find σ_x, σ_y in Terms of α_0, α_1

$$\sigma_x = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_1) \qquad \sigma_y = \frac{1}{\sqrt{2}}(\alpha_0 - \alpha_1)$$

$$|\psi_a\rangle = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_1)|x\rangle + \frac{1}{\sqrt{2}}(\alpha_0 - \alpha_1)|y\rangle$$

Quantum Observables/Operators

- ***Observable*** is an Attribute of Physical System
- In Principle, an Observable can be ***Measured***
- In QM, Observable is Associated with a ***Hermitian (self-adjoint) Operator***
- Measured Value is ***Eigenvalue*** of Operator Matrix

Hilbert Space Operators

- Operator $\mathbf{U}$ in Hilbert Space $\mathbb{H}^n$ is:

$$\begin{array}{ll} \textit{Hermitian (self - adjoint)} & \text{if } \mathbf{U} = \mathbf{U}^\dagger \\ \textit{Unitary} & \text{if } \mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I} \\ \textit{Normal} & \text{if } \mathbf{U}\mathbf{U}^\dagger - \mathbf{U}^\dagger\mathbf{U} = 0 \end{array}$$

- Operator Maps State Vectors to Different States, Mathematically Modeled as:

$$|\psi_b\rangle = \mathbf{U}|\psi_a\rangle$$

- Note that:

$$\mathbf{U}(a|\psi_a\rangle + b|\psi_b\rangle) = a\mathbf{U}|\psi_a\rangle + b\mathbf{U}|\psi_b\rangle$$

Hilbert Space Operators

- α_i are the State Amplitudes of the State Vector:

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle ... + \alpha_i|i\rangle ... + \alpha_{n-1}|n-1\rangle$$

- α_i can be calculated as:

$$\alpha_j = \langle j|\psi_a\rangle, \forall j = 0,1,...,n-1$$

- Note that:

$$|\psi_b\rangle = \mathbf{U}|\psi_a\rangle$$

$$\langle j|\psi_b\rangle = \langle j|\mathbf{U}|\psi_a\rangle$$

Projection Operator Construction

- Consider Hilbert Space $\mathbb{H}^2$ with Basis:

$$\{|0\rangle, |1\rangle\}$$

- Determine Operator $\mathbf{U}$ to Interchange Projection between Basis Vectors:

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle \mapsto \alpha_1 |0\rangle + \alpha_0 |1\rangle$$

- $\mathbf{U}$ is Defined as:

$$\mathbf{U} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\mathbf{U} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Projection Operator Example

- Consider the Quantum State:

$$|\psi_a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$|\psi_b\rangle = \mathbf{U} |\psi_a\rangle$$

$$\begin{aligned} \mathbf{U} |\psi_a\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha_0 |0\rangle + \alpha_1 |1\rangle) \\ &= \alpha_0[(|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle] \\ &\quad + \alpha_1[(|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle] \end{aligned}$$

- We know that:

$$\langle 0|1\rangle = \langle 1|0\rangle = 0 \qquad \langle 0|0\rangle = \langle 1|1\rangle = 1$$

Proj. Operator Example (cont.)

$$\mathbf{U}|\psi_a\rangle = \alpha_0[(|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle] + \alpha_1[(|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle]$$

$$\mathbf{U}|\psi_a\rangle = \alpha_0(|0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle) + \alpha_1(|0\rangle\langle 1|1\rangle + |1\rangle\langle 0|1\rangle)$$

$$\mathbf{U}|\psi_a\rangle = \alpha_0|1\rangle\langle 0|0\rangle + \alpha_1|0\rangle\langle 1|1\rangle$$

$$\mathbf{U}|\psi_a\rangle = \alpha_0|1\rangle + \alpha_1|0\rangle$$

Video: The Qubit

Projection Operators - Projectors

- Outer Product of State Vector with Itself Yields a Projection Operator:

$$|\psi_a\rangle\langle\psi_a| = \mathbf{P}_{\psi_a}$$

- Property:

$$(\mathbf{P}_{\psi_a})^2 = |\psi_a\rangle\langle\psi_a|\psi_a\rangle\langle\psi_a| = |\psi_a\rangle\langle\psi_a| = \mathbf{P}_{\psi_a}$$

- Orthogonality Definition:

$$\mathbf{P}_i\mathbf{P}_j|\psi_a\rangle = 0$$

- Often Written as:

$$\mathbf{P}_i\mathbf{P}_j = 0$$

Rotation Operator

- Produces new Quantum State that is a Coordinate Rotation of Current State
- Spin 1/2 about Z-axis Rotations
- Fermions (e^-, protons)
- Basis States:

$$|+\rangle \text{ "spin up" } \quad \text{spin number is } s = +1/2$$
$$|-\rangle \text{ "spin down" } \quad \text{spin number is } s = -1/2$$

$$\begin{array}{ccc} & & \begin{array}{cc} |+\rangle & |-\rangle \end{array} \\ \langle + | \longrightarrow & & \\ & \mathbf{R}_z(\phi) = & \begin{bmatrix} e^{+i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{bmatrix} \\ \langle - | \longrightarrow & & \end{array}$$

Rotation Operator

- Produces new Quantum State that is a Coordinate Rotation of Current State
- Integer Spin about Z-axis Rotation
- Bosons (photons)
- Basis States:

$|+\rangle$ spin number is $s=+1$ $\quad$ $|-\rangle$ spin number is $s=-1$

$|0\rangle$ spin number is $s=0$

$$\mathbf{R}_z(\phi)=\begin{array}{c|ccc} & |+\rangle & |0\rangle & |-\rangle \\ \hline \langle +| & e^{+i\phi} & 0 & 0 \\ \langle 0| & 0 & 1 & 0 \\ \langle -| & 0 & 0 & e^{-i\phi} \end{array}$$

Rotation Operator

- Produces new Quantum State that is a Coordinate Rotation of Current State
- RHC/LHC Polarization of photon
- Basis States:

$$|R\rangle=\frac{1}{\sqrt{2}}(|x\rangle+i|y\rangle) \quad m=+1 \;\; \text{(RHC polarized)}$$

$$|L\rangle=\frac{1}{\sqrt{2}}(|x\rangle-i|y\rangle) \quad m=-1 \;\; \text{(LHC polarized)}$$

$$\mathbf{R}_z(\phi)=\begin{array}{c|cc} & |R\rangle & |L\rangle \\ \hline \langle R| & e^{+i\phi} & 0 \\ \langle L| & 0 & e^{-i\phi} \end{array}$$

Spectral Decomposition

- Spectral Decomposition of an Operator is Representation of Operator as Linear Combination of Projectors
- Eigenvalues of Operator are Coefficients of Projectors in Linear Combination
- Recall Eigenvalue: $\mathbf{U}\,|\,\psi\rangle = \lambda\,|\,\psi\rangle$

$$\mathbf{I}\,|\,\psi\rangle = 1\,|\,\psi\rangle$$

$$\mathbf{U}\,|\,\psi\rangle = \lambda\mathbf{I}\,|\,\psi\rangle$$

$$(\mathbf{U}-\lambda\mathbf{I})\,|\,\psi\rangle = 0$$

Spectral Decomposition

- Let the Following be Orthonormal Basis in n-dim Hilbert Space:

$$\{|\,e_0\rangle,|\,e_1\rangle,\ldots,|\,e_i\rangle,\ldots,|\,e_{n-1}\rangle\}$$

- Let $\mathbf{U}$ be a normal operator and:

$$|\,\psi\rangle = \sum_{i=0}^{n-1}\gamma_i\,|\,e_i\rangle \qquad (\mathbf{U}-\lambda\mathbf{I})\sum_{i=0}^{n-1}\gamma_i\,|\,e_i\rangle = 0$$

$$\mathbf{U} = [u_{ij}] \qquad \mathbf{I} = [\delta_{ij}]$$

non-trivial Soln iff:

$$\sum_{i=0}^{n-1}(u_{ij}-\lambda\delta_{ij})\gamma_i = 0 \longrightarrow \det(\mathbf{U}-\lambda\mathbf{I}) = 0$$

Observable

- ***Observable*** is any Hermitian Operator whose Eigenvectors form a Basis:
- Facts about Measurement Operators:

1. Eigenvalues of Hermitians are Real
2. Eigenvectors corresponding to different Eigenvalues are Orthogonal
3. If 2 Hermitians Commute-common basis of orthonormal Eignvectors an Eigenbasis
4. ***Complete Set of commuting Observables*** Defined as Minimal Set of Hermitians with Unique Common Eigenbasis

Hermitian Eigenvalue

Let $|\phi\rangle$ be a unit eigenvector (eigenket) of the hermitian matrix $\mathbf{U}$

$$\mathbf{U}|\phi\rangle = \lambda|\phi\rangle$$

Take the adjoint of both sides of this equation

$$(\mathbf{U}|\phi\rangle)^\dagger = (\lambda|\phi\rangle)^\dagger$$

$$\langle\phi|\mathbf{U}^\dagger = \lambda^*\langle\phi|$$

Since $\mathbf{U}$ is hermitian :

$$\mathbf{U}^\dagger = \mathbf{U}$$

$$\langle\phi|\mathbf{U} = \lambda^*\langle\phi|$$

Hermitian Eigenvalue (cont)

$$\langle\phi|\mathbf{U} = \lambda^*\langle\phi|$$

Multiply both sides of equation by eigenket $|\phi\rangle$

$$\underbrace{\langle\phi|\mathbf{U}|\phi\rangle} = \lambda^*\langle\phi|\phi\rangle$$

Definition of eigenket

$$\mathbf{U}|\phi\rangle = \lambda|\phi\rangle$$

Inner product of eigenket with itself is “1”

$$\langle\phi|\lambda|\phi\rangle = \lambda^*$$

$$\lambda\langle\phi|\phi\rangle = \lambda^*$$

$$\lambda = \lambda^*$$

Thus, λ must be a real value

Normal Operator

- Recall that a Normal Operator is one that:

$$\mathbf{N}^\dagger\mathbf{N} = \mathbf{N}\mathbf{N}^\dagger$$

- Every Normal Operator has Complete set of Orthonormal Eigenvectors

$$\mathbf{N}|n_i\rangle = \lambda_i|n_i\rangle$$

- Every State Vector can be Expressed Using the Basis formed by the n Eigenvectors of a Normal Operator $\mathbf{N}$

$$|\psi_a\rangle = \sum_{i=0}^{n-1}\alpha_i|n_i\rangle$$

Normal Operator (cont)

- Normality of **N** Implies:

$$\sum_{i=0}^{n-1} | \alpha_i |^2 = 1$$

- Thus,

$$\mathbf{N} | \psi_a \rangle = \mathbf{N} \sum_{i=0}^{n-1} \alpha_i | n_i \rangle = \sum_{i=0}^{n-1} \alpha_i \mathbf{N} | n_i \rangle = \sum_{i=0}^{n-1} \alpha_i \lambda_i | n_i \rangle$$

- Outer Product of State Vector with itself is Projection Operator:

$$| \psi_a \rangle \langle \psi_a | = \mathbf{P}_{\psi_a}$$

Normal Operator (cont)

- Constructing Projection Operator using Eigenvectors of **N**:

$$\mathbf{P}_i = | n_i \rangle \langle n_i |$$

- Applying this Projection to a State Vector:

$$\mathbf{P}_i | \psi_a \rangle = | n_i \rangle \langle n_i | \sum_{j=0}^{n-1} \alpha_j | n_j \rangle = \sum_{j=0}^{n-1} \alpha_j | n_i \rangle (\langle n_i | n_j \rangle)$$

- Kronecker Delta function Occurs:

$$\langle n_i | n_j \rangle = \delta_{ij}$$

SIFTING PROPERTY

$$\mathbf{P}_i | \psi_a \rangle = \alpha_i | n_i \rangle$$

Normal Operator (cont)

- Substituting this in Earlier Result:

$$\mathbf{P}_i \,|\psi_a\rangle = \alpha_i \,|n_i\rangle$$

$$\mathbf{N}\,|\psi_a\rangle = \sum_{i=0}^{n-1} \alpha_i \lambda_i \,|n_i\rangle = \sum_{i=0}^{n-1} \lambda_i \mathbf{P}_i \,|\psi_a\rangle$$

- Leading to the Interesting Result:

$$\mathbf{N} = \sum_{i=0}^{n-1} \lambda_i \mathbf{P}_i$$

- Spectral Decomposition of **N** is Independent of Basis

Spectral Decomposition Example

- Consider a Normal Operator,**N**, in $\mathbb{H}^2$ with eigenvalues λ_a and λ_b
- Corresponding Orthonormal Eigenstates Characterized by Eigenvectors:

$$|a\rangle = \alpha_0 \,|0\rangle + \alpha_1 \,|1\rangle \qquad |b\rangle = \beta_0 \,|0\rangle + \beta_1 \,|1\rangle$$

- Corresponding Projection Operators:

$$\mathbf{P}_a = |a\rangle\langle a| = \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 \end{bmatrix} \quad \mathbf{P}_b = |b\rangle\langle b| = \begin{bmatrix} |\beta_0|^2 & \beta_0\beta_1^* \\ \beta_1\beta_0^* & |\beta_1|^2 \end{bmatrix}$$

Spectral Decomposition Example

- Corresponding Projection Operators:

$$\mathbf{P}_a = |a\rangle\langle a| = \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 \end{bmatrix} \quad \mathbf{P}_b = |b\rangle\langle b| = \begin{bmatrix} |\beta_0|^2 & \beta_0\beta_1^* \\ \beta_1\beta_0^* & |\beta_1|^2 \end{bmatrix}$$

- We can use the Spectral Decomposition to Write the Operator as:

$$\mathbf{N} = \lambda_a \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 \end{bmatrix} + \lambda_b \begin{bmatrix} |\beta_0|^2 & \beta_0\beta_1^* \\ \beta_1\beta_0^* & |\beta_1|^2 \end{bmatrix}$$

Measurement of Observables

- Numerical Outcome of Measurement is an Eigenvalue of the Operator
- Immediately after Measurement, Quantum State is Eigenstate (an eigenvector of the Operator)
- Spectral Decomposition of Operator specifies Exhaustive Measurement in Sense that all Possible Outcomes (the eigenvalues) are Specified
- Result is (pp.70-71 Marinescu):

$$\text{Prob}(\lambda_x \mid \psi_a\rangle) = \langle\psi_a \mid \mathbf{P}_x \mid \psi_a\rangle = |\alpha_x|^2$$

Observable Summary

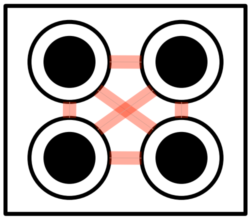
- Observable thought of as specific question posed to a quantum system (eg. What is the position of a photon after passing through a beam splitter?)
- Mathematical analog is corrspondence of a hermitian operator
- Eigenvalues of hermitian operator are real
  - eigenvalues are only possible values observable can take as a result of measuring it on any given state
  - eigenkets of observable form a basis for the quantum state

Observable Summary (cont.)

- Observable thought of as specific question posed to a quantum system (eg. What is the position of a photon after passing through a beam splitter?)
- Observable is a question
- Question has a SET of possible answers
- The set of possible answers are the eigenvalues of the observable
- If the expected value of an extemely large set of observables (either over time OR over all instances in the multiverse – ergodicity) is an eigenvalue, then observable is “SHARP”

Observable Example (cont)

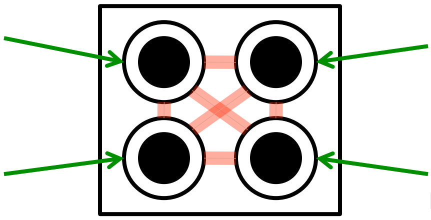
- Consider a QCA cell with a single electron and four Quantum dots
- Assume the probability of tunneling is very high since the electron has a lot of energy



- Assume tunneling probability is equal among all quantum dots (tunnels denoted by red lines)
- Let measurement of interest be which of the quantum wells contains the electron

Observable Example

- Let each Q-dot represent a basis state denoted as:



$$|x_0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |x_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|x_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |x_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Quantum State of electron is:

$$|\psi\rangle = \alpha_0 |x_0\rangle + \alpha_1 |x_1\rangle + \alpha_2 |x_2\rangle + \alpha_3 |x_3\rangle$$

$$\text{Prob}[|\psi\rangle = |x_i\rangle] = |\alpha_i|^2$$

Observable Example

- Let observable be denoted as:

$$P(|\psi\rangle) = P\left(\sum_{i=0}^{3} \alpha_i |x_i\rangle\right)$$

- Now Construct the Observable Hermitian:

$$\mathbf{P} = \sum_{i=0}^{3} \alpha_i |x_i\rangle\langle x_i| \alpha_i^* = \sum_{i=0}^{3} \alpha_i \alpha_i^* |x_i\rangle\langle x_i|$$

$$\mathbf{P} = \sum_{i=0}^{3} |\alpha_i|^2 |x_i\rangle\langle x_i|$$

Observable Example

$$\mathbf{P} = \sum_{i=0}^{3} |\alpha_i|^2 |x_i\rangle\langle x_i|$$

$$\mathbf{P} = |\alpha_0|^2 |x_0\rangle\langle x_0| + |\alpha_1|^2 |x_1\rangle\langle x_1| + |\alpha_2|^2 |x_2\rangle\langle x_2| + |\alpha_3|^2 |x_3\rangle\langle x_3|$$

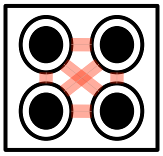
$$\mathbf{P} = |\alpha_0|^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + |\alpha_1|^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ |\alpha_2|^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + |\alpha_3|^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the spectral decomposition of the observable P

A linear combination of Normal Projectors

Observable Example



$$\mathbf{P} = \begin{bmatrix} |\alpha_0|^2 & 0 & 0 & 0 \\ 0 & |\alpha_1|^2 & 0 & 0 \\ 0 & 0 & |\alpha_2|^2 & 0 \\ 0 & 0 & 0 & |\alpha_3|^2 \end{bmatrix}$$

- The eigenvalues of this observable are:

$$\lambda_0 = |\alpha_0|^2, \lambda_1 = |\alpha_1|^2, \lambda_2 = |\alpha_2|^2, \lambda_3 = |\alpha_3|^2$$

- Measurement using this observable forces the quantum state to evolve into an eigenket:

$$|x_i\rangle$$

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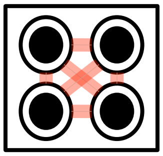

$$\mathbf{P} = \begin{bmatrix} |\alpha_0|^2 & 0 & 0 & 0 \\ 0 & |\alpha_1|^2 & 0 & 0 \\ 0 & 0 & |\alpha_2|^2 & 0 \\ 0 & 0 & 0 & |\alpha_3|^2 \end{bmatrix}$$

- The expectation that a particular eigenket is observed after applying the observable is:

$$|\alpha_i|^2 = \frac{1}{4}$$

- This is because the tunneling probablities are all equal and very high among the four Q-dots
- These are also the eigenvalues of the observable – it is NOT sharp