ECE/CS 5/7384 Introduction to Quantum Computing
Homework 1

<u>General instructions</u>: This homework is intended to partially test your comprehension of the mathematics topics that are commonly used to understand quantum computing algorithms. For this reason, you should do all these calculations manually without relying on computer-aided computations (*e.g.*, either symbolic tools such as Wolfram alpha, Mathematica, *etc*. or other mathematical programming tools such as MatLab OR using large language models like Chat GPT). If you do not follow these instructions and choose to cheat anyway, you may get a good score on this homework assignment, but you will not do well on the exam which is more heavily weighted in your final grade for the class as you will have to use these types of calculations to answer exam questions – so consider yourself forewarned. For full credit, you must show all intermediate steps in your calculations no matter how trivial they seem to you. Showing all intermediate details helps me to understand your thought process and to help you find any mistakes that you might make. I will not give credit for just writing the solution to the problem without showing how it was derived or calculated. You should also use Dirac's notation for linear algebraic calculations rather than explicit notation where possible unless I ask you otherwise. Dirac's notation, also known as "BraKet" notation, is the language of quantum computing, hence a goal of this class is to become comfortable using it.

1. Consider the transfer matrices for the following single qubit gates.

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$\mathbf{V} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$

For all of the following questions, express your answer in simplified form for full credit. Simplified form means your answer should have as few terms as possible. You may use the conjugate transpose of a matrix in your answer. As a reminder, the conjugate transpose of a matrix $\mathbf{A}$ is denoted as $\mathbf{A}^\dagger$.

a) Give an expression for $\mathbf{H}$ in terms of $\mathbf{X}$ and $\mathbf{Z}$ only.

b) Give an expression for **Y** in terms of **X** and **Z** only.

c) Give an expression for **X** in terms of **H** and **Z** only.

d) Show that $\mathbf{V}^\dagger$ is a square root of **X**.

e) Find an expression for **Y** in terms of **S** and **X**.

f) Find an expression for **X** in terms of **S** and **Y**:

1A. **This question is for 7384 students only, no credit will be awarded if you are enrolled in CS 5384 or ECE 5384.**

Find an expression for $\mathbf{H}$ in terms of $\mathbf{S}$ and $\mathbf{V}$.

HINT: You can multiply the expression in your answer that contains $\mathbf{S}$ and $\mathbf{V}$ by a global phase factor in the form of $e^{i\theta}$ where θ is some constant phase angle in radians. In other words, it is permissible for your answer to be in the form of $\mathbf{H} = e^{i\theta}(REMINAING\ EXPRESSION\ IN\ TERMS\ OF\ \mathbf{S}\ AND\ \mathbf{V})$ where the angle, θ, is some constant value.

2. Consider the following kets (*i.e.*, column vectors).

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix},\ |1\rangle = \begin{bmatrix}0\\1\end{bmatrix},\ |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}},\ |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}},\ |\circlearrowright\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}},\ |\circlearrowleft\rangle = \frac{|0\rangle-i|1\rangle}{\sqrt{2}}$$

a) Show that $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$ can serve as an orthonormal basis set for $\mathbb{H}_2$ where $\mathbb{H}_2$ denotes a two-dimensional complex Hilbert vector space.

b) Compute the following values using Dirac's notation with respect to the computational basis pair, $|0\rangle$ and $|1\rangle$. The answers may be either scalars, vectors or matrices. All answers should be expressions in terms of the computational basis set $\{|0\rangle, |1\rangle\}$ in Dirac's notation. Dirac's notation is sometimes referred to "BraKet" notation.

$i)\ \langle ↺ |+\rangle = ??$

$ii)\ |0\rangle\langle +| = ??$

$iii)\ \langle +|-\rangle = ??$

$iv)\ |↻\rangle\langle -| = ??$

$v)\ |+ -\rangle = ??$

$vi)\ |{+}{-}\circlearrowleft\rangle =??$

$vii)\ |{+}\circlearrowleft\rangle\langle 1| =??$

c) For each of your answers to Question 2, part b) that are reproduced below (*i.e.*, answers to parts i) through vii) above), rewrite the answer in explicit notation instead of Dirac's notation. That is, give scalars as numerical quantities (some may be complex values), give vectors as explicit row or column vectors using the notation from your linear algebra class, and give each matrix as an array of values, also similar to your usage in your linear algebra class.

$i)\ \langle\circlearrowleft|+\rangle = ??$

$ii)\ |0\rangle\langle+| = ??$

$iii)\ \langle+|-\rangle = ??$

$iv)\ |↻\rangle\langle-| = ??$

$v)\ |+-\rangle = ??$

$vi)\ |{+}{-}\circlearrowleft\rangle = ??$

$vii)\ |{+}\circlearrowleft\rangle\langle 1| = ??$

3. A particular set of well-known quantum state transformations are the Pauli matrices denoted by **X**, **Y** and **Z**. These are also known as the Pauli-**X**, Pauli-**Y** and Pauli-**Z** matrices, respectively. The Pauli matrices are defined as

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

a) Express **X**, **Y** and **Z** as expressions using Dirac's notation in terms of the computational basis set $\{|0\rangle, |1\rangle\}$. You can use the complex transpose of the computational basis vectors, $\{\langle 0|, \langle 1|\}$, in your answer also. Show all steps in your derivation, no credit for "guessing" the right answer.

b) Calculate **I**, $\mathbf{X}^2$, $\mathbf{Y}^2$ and $\mathbf{Z}^2$ using Dirac's notation and also show that $\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = \mathbf{I}$. Show all steps in your derivation, no credit for "guessing" the right answer.

c) Show that the Pauli spin matrices **X**, **Y** and **Z**, are unitary. Show all steps in your derivations, no credit for "guessing" the right answer.

d) Find a constant scalar value c_1 such that $c_1\mathbf{XYZ} = \mathbf{I}$. For full credit, use Dirac's notation in all calculations. Show all steps in your derivation, no credit for "guessing" the right answer.

e) Express **X** as a function of **Y**, **Z** and a scalar constant d_1. You must give the exact value of the scalar d_1 for full credit. Show all steps in your derivation, no credit for "guessing" the right answer.

f) Express **Y** as a function of **X**, **Z** and a scalar constant d_2. You must give the exact value of the scalar d_2 for full credit. Show all steps in your derivation, no credit for "guessing" the right answer.

g) Express **Z** as a function of **X**, **Y** and a scalar constant d_3. You must give the exact value of the scalar d_3 for full credit. Show all steps in your derivation, no credit for "guessing" the right answer.

h) In part d), you found the scalar constant c_1 that satisfies $c_1\mathbf{XYZ} = \mathbf{I}$. There exist similar relationships for the other five expressions that are products of the three distinct Pauli spin matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. These five expressions are $c_2\mathbf{XZY} = \mathbf{I}$, $c_3\mathbf{YXZ} = \mathbf{I}$. $c_4\mathbf{YZX} = \mathbf{I}$, $c_5\mathbf{ZXY} = \mathbf{I}$ and $c_6\mathbf{ZYX} = \mathbf{I}$. Derive these five relationships and give the values for the scalar constants. Show all steps in your derivation, no credit for "guessing" the right answer.

4. Consider qubit operators represented by the **X**, **Y**, **Z** and **H** matrices.

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

a) Find the characteristic equation and eigenvalues for the Pauli-**X** operator. Show details for your calculations.

b) Find the characteristic equation and eigenvalues for the Pauli-**Y** operator. Show details for your calculations.

c) Find the characteristic equation and eigenvalues for the Pauli-**Z** operator. Show details for your calculations.

d) Find the characteristic equation and eigenvalues for the Hadamard operator, $\mathbf{H}$. Show details for your calculations.

5. **This question is for 7384 students only, no credit will be awarded if you are enrolled in CS 5384 or ECE 5384.**

Consider a 2×2 matrix, **A**:

$$\mathbf{A} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

a) Is **A** a *Hermitian* matrix? Give the mathematical justification for your answer of "yes" or "no" by either showing that **A** is or is not Hermitian. (note that Hermitian matrices are sometime referred to as "self-adjoint" matrices – this means the same thing).

b) Is **A** a *normal* matrix? Give the mathematical justification for your answer of "yes" or "no" by either showing that **A** is or is not normal.

c) Is **A** a *unitary* matrix? Give the mathematical justification for your answer of "yes" or "no" by either showing that **A** is or is not unitary.

d) Find the eigenvalues of the **A** matrix. Show the characteristic polynomial and solve for its roots, do not just write down the eigenvalues.

e) Find the eigenvectors of the **A** matrix. Show all calculations for full credit.