Deutsch's Problem

- Black Box Characterized by Transfer Function that Maps Single input Bit x to Output Bit $f(x)$
- Transformation Performed by Black Box Could be Any of the Four Possible Boolean Functions of One Variable (pick a pair of $f(0)$ and $f(1)$):

$$f(0)=0 \qquad f(0)=1 \qquad f(1)=0 \qquad f(1)=1$$

- Problem Posed by Professor David Deutsch is to Distinguish if:

$$f(0)=f(1) \qquad \text{OR} \qquad f(0)\neq f(1)$$

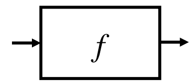
Deutsch's Problem

- Four cases for $f(x)$:

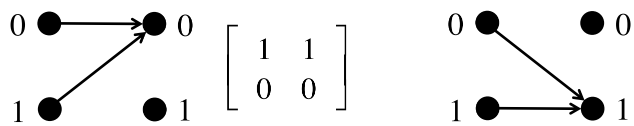
Unbalanced: $f(0)=f(1)$

Balanced: $f(0)\neq f(1)$

Deutsch's Problem

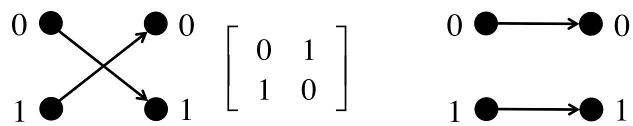


- What are the transfer matrices for $f(x)$?



$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Unbalanced: $f(0) = f(1)$

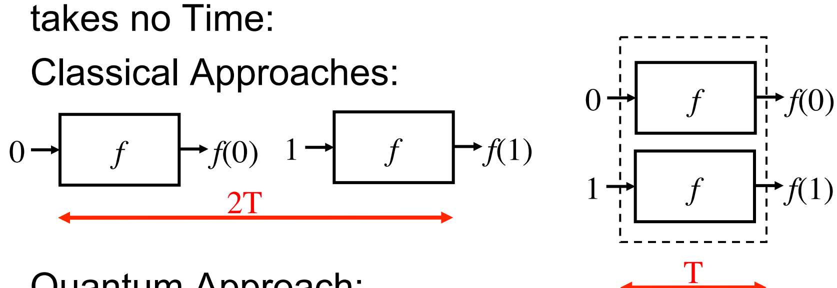


$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

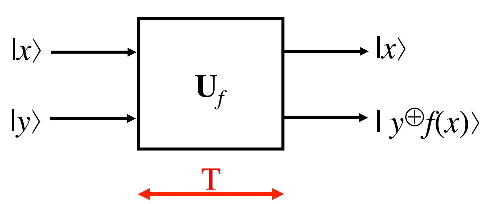
Balanced: $f(0) \neq f(1)$

Deutsch's Problem

- Assume Comparison of Function Evaluation takes no Time:
- Classical Approaches:

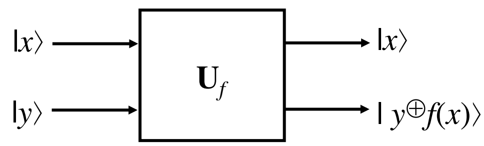


- Quantum Approach:



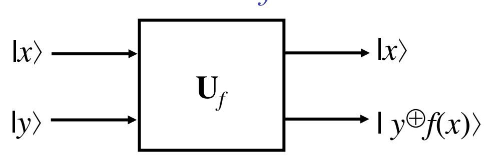
Deutsch's Problem

- $f(x)$ is "embedded" inside $\mathbf{U}_f$ "gate" or operator but the actual $f(x)$ embedded is unknown:
- Denoted by:



- There are four Different $\mathbf{U}_f$ gates, but it is unknown which is present

Synthesis of $\mathbf{U}_f$ Operators



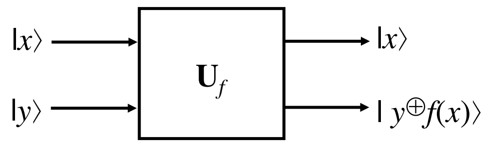
- Use the following notation:

$$f_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad f_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad f_{01} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad f_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Truth tables for $\mathbf{U}_f$:

x	y	f_0	$y \oplus f_0$	f_1	$y \oplus f_1$	f_{01}	$y \oplus f_{01}$	f_{I}	$y \oplus f_{\mathrm{I}}$
0	0	0	0	1	1	1	1	0	0
0	1	0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

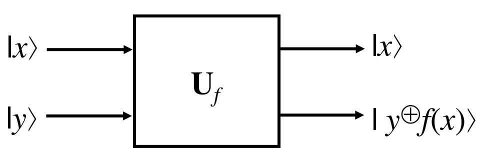
Synthesis of $\mathbf{U}_{f0}$



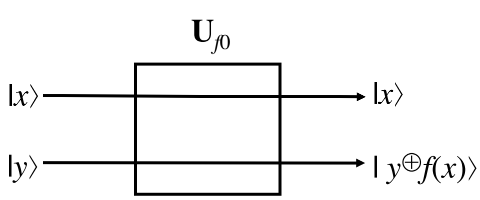
x	y	f_0	$y \oplus f_0$	f_1	$y \oplus f_1$	f_{01}	$y \oplus f_{01}$	f_I	$y \oplus f_I$
0	0	0	0	1	1	1	1	0	0
0	1	0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

$$\mathbf{U}_{f0} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Synthesis of $\mathbf{U}_{f0}$

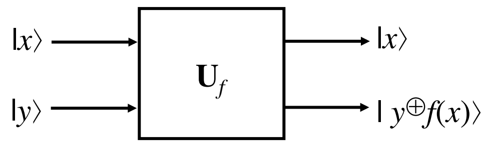


$$\mathbf{U}_{f0} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$



$$\mathbf{U}_{f0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

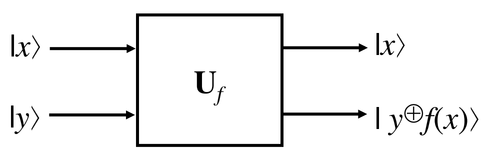
Synthesis of $\mathbf{U}_{f1}$



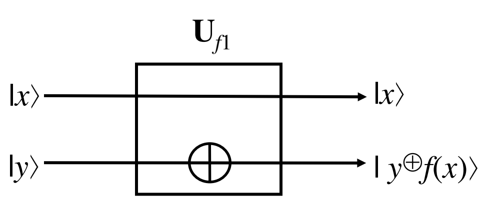
x	y	f_0	$y \oplus f_0$	f_1	$y \oplus f_1$	f_{01}	$y \oplus f_{01}$	f_I	$y \oplus f_I$
0	0	0	0	1	1	1	1	0	0
0	1	0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

$$\mathbf{U}_{f1} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 1 & 0 & 0 \\ 01 & 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

Synthesis of $\mathbf{U}_{f1}$

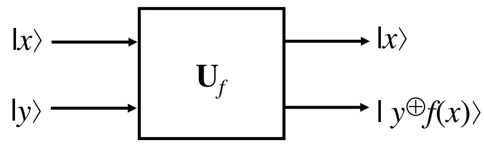


$$\mathbf{U}_{f1} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 1 & 0 & 0 \\ 01 & 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$



$$\mathbf{U}_{f1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

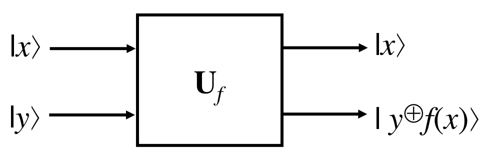
Synthesis of $\mathbf{U}_{fI}$



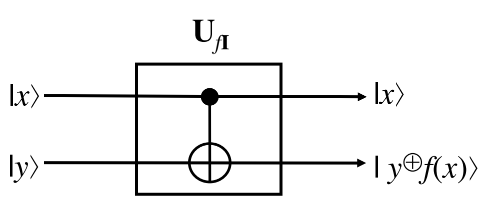
x	y	f_0	$y \oplus f_0$	f_1	$y \oplus f_1$	f_{01}	$y \oplus f_{01}$	f_I	$y \oplus f_I$
0	0	0	0	1	1	1	1	0	0
0	1	0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

$$\mathbf{U}_{fI} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Synthesis of $\mathbf{U}_{fI}$

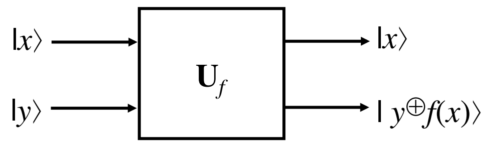


$$\mathbf{U}_{fI} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



Controlled-NOT Gate

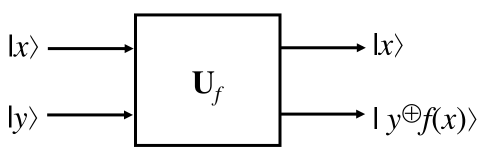
Synthesis of $\mathbf{U}_{f01}$



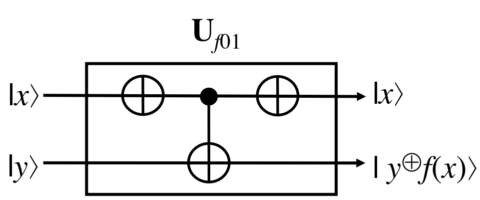
x	y	f_0	$y\oplus f_0$	f_1	$y\oplus f_1$	f_{01}	$y\oplus f_{01}$	f_{I}	$y\oplus f_{\mathrm{I}}$
0	0	0	0	1	1	1	1	0	0
0	1	0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	1	0	0	1	1	0

$$\mathbf{U}_{f01} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Synthesis of $\mathbf{U}_{f01}$



$$\mathbf{U}_{f01} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right) \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Synthesis of $\mathbf{U}_{f01}$ (cont)

$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

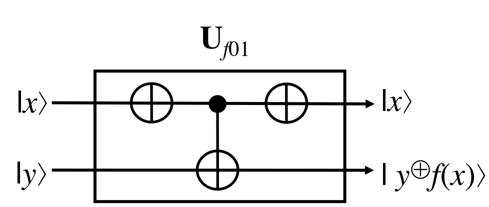
$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

Synthesis of $\mathbf{U}_{f01}$ (cont)

$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\mathbf{U}_{f01} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$$



Quantum Algorithm

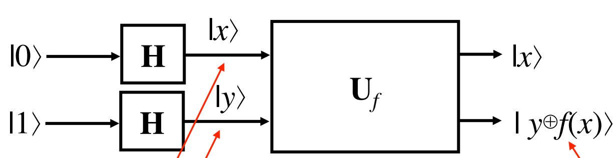
- Initialize Qubits to known state
- To exploit parallelism, place initialized qubits into a state of superposition
- Apply a series of evolutions of the quantum state of the system. This is accomplished by applying a series of quantum unitary operations to the quantum state
- Measure the quantum state – forcing it to collapse into an eigenket
- For deterministic output, the observable must be SHARP

Deutsch's Problem

- Initially Place the Input Qubits into the Following States of Superposition:

  $|x\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ $\qquad$ $|y\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$

- This is Accomplished via a Hadamard Transform (or the use of 2 Hadamard Gates):



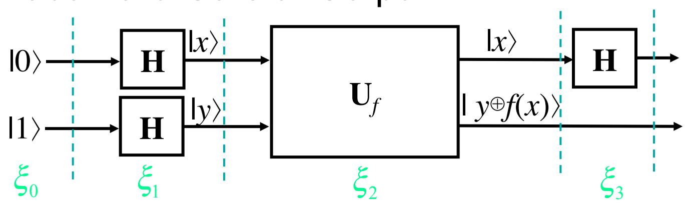
$$\mathbf{H}\,|0\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$$

$$\mathbf{H}\,|1\rangle = (1/\sqrt{2})(|0\rangle - |1\rangle)$$

As in Previous Example, Outputs are in Superposition State. Measurement here Yields Differing Results Probabilistically

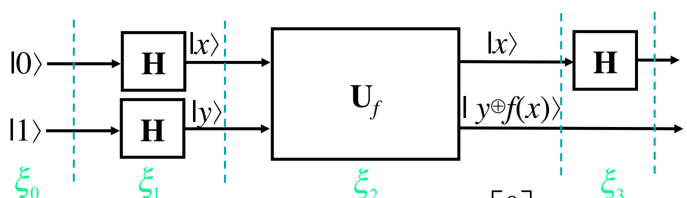
Deutsch's Problem

- Must "Force" Output to Desired Basis State for a Deterministic Circuit/Algorithm to Result
- Accomplished by Placing an Additional Hadamard Gate at Output



- We Analyze this Quantum System (Circuit/ Algorithm) Stage-by-Stage

Deutsch's Problem



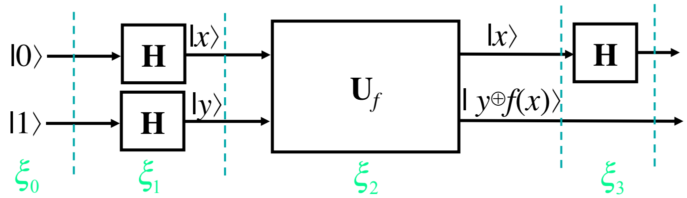
XI
KS-EYE

$$|\xi_0\rangle = |01\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

- Transfer Matrix of First Stage:

$$\mathbf{G}_1 = \mathbf{H} \otimes \mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{bmatrix}$$

Deutsch’s Problem



$$|\xi_1\rangle = |xy\rangle = \mathbf{G}_1 |01\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$|\xi_1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) = [(|0\rangle + |1\rangle)/\sqrt{2}][(|0\rangle - |1\rangle)/\sqrt{2}]$$

Deutsch’s Problem

- Note that:

$$|y \oplus f(x)\rangle$$

- and,

$$|y\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

- thus,

$$|y \oplus f(x)\rangle = |(|0\rangle - |1\rangle)/\sqrt{2} \oplus f(x)\rangle$$
$$= |(|0\rangle \oplus f(x) - |1\rangle \oplus f(x))/\sqrt{2}\rangle$$

- using the fact:

$$|0 \oplus f(x)\rangle = |f(x)\rangle$$

$$|y \oplus f(x)\rangle = (|f(x)\rangle - |1 \oplus f(x)\rangle)/\sqrt{2}$$

Deutsch's Problem

- In Problem Statement we Know that:

$$f(x)=1 \qquad \text{OR} \qquad f(x)=0$$

- thus:

$$|1\oplus f(x)\rangle = \begin{cases} |0\rangle, & f(x)=1 \\ |1\rangle, & f(x)=0 \end{cases}$$

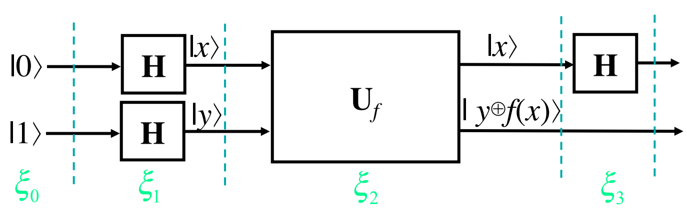
- yielding,

$$|y\oplus f(x)\rangle = (|f(x)\rangle - |1\oplus f(x)\rangle)/\sqrt{2}$$

$$|y\oplus f(x)\rangle = \begin{cases} (|0\rangle - |1\rangle)/\sqrt{2}, & f(x)=0 \\ -(|0\rangle - |1\rangle)/\sqrt{2}, & f(x)=1 \end{cases}$$

$$|y\oplus f(x)\rangle = (-1)^{f(x)}(|0\rangle - |1\rangle)/\sqrt{2}$$

Deutsch's Problem



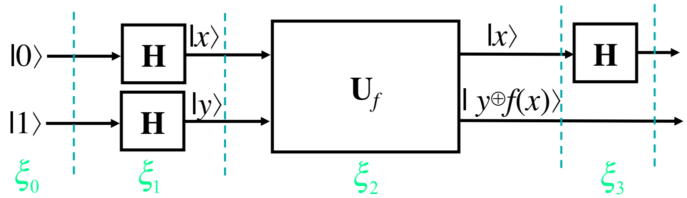
- When $f(0)=f(1)=0$:

$$|\xi_2\rangle = |x, y\oplus f(x)\rangle = |x\rangle \otimes |y\oplus f(x)\rangle$$

$$|x\rangle = \mathbf{H}|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|y\oplus f(x)\rangle = \frac{|f(x)\rangle - |1\oplus f(x)\rangle}{\sqrt{2}}$$

Deutsch's Problem



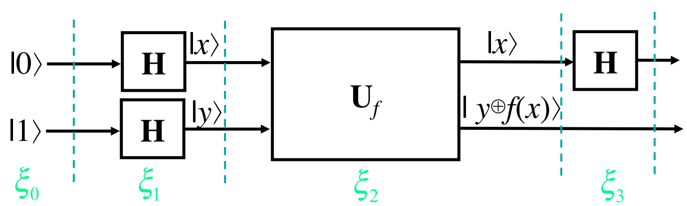
- Since $f(x)=0$:

$$|y \oplus f(x)\rangle = \frac{|f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\xi_2\rangle = |x\rangle \otimes |y \oplus f(x)\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\xi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Deutsch's Problem

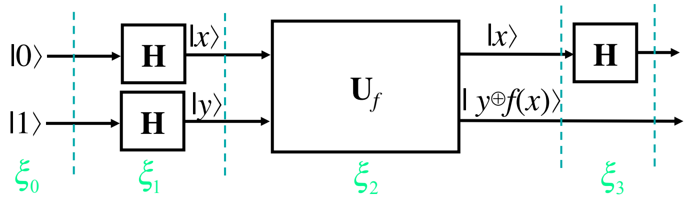


- Since $f(x)=0$:

$$|\xi_2\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

$$|\xi_2\rangle = \frac{1}{2}\left(|00\rangle - |11\rangle\right) = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Deutsch's Problem

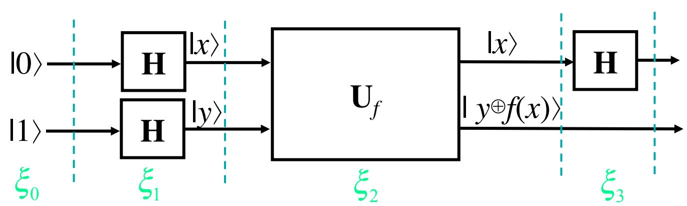


- When $f(0)=f(1)$:

$$|\xi_2\rangle = \begin{cases} \left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=f(1)=0 \\ -\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=f(1)=1 \end{cases}$$

Obtained using similar derivation as that for $f(x)=0$

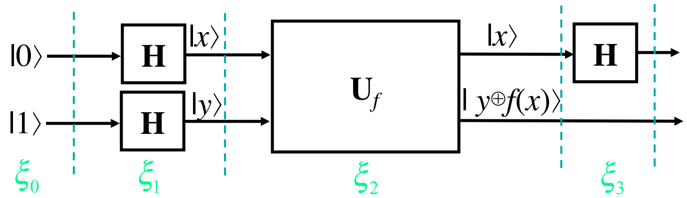
Deutsch's Problem



- When $f(0)=f(1)$:

$$|\xi_2\rangle = \begin{cases} \frac{1}{2}\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T, & f(0)=f(1)=0 \\ -\frac{1}{2}\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T, & f(0)=f(1)=1 \end{cases}$$

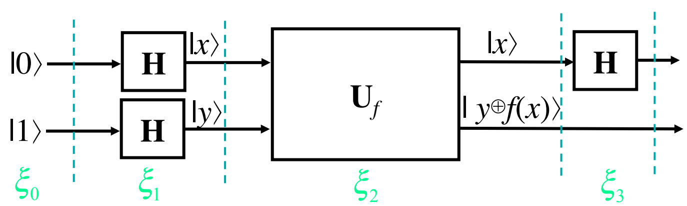
Deutsch's Problem



- When $f(0)\neq f(1)$:

$$|\xi_2\rangle = \begin{cases} \left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=0 \text{ and } f(1)=1 \\ -\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=1 \text{ and } f(1)=0 \end{cases}$$

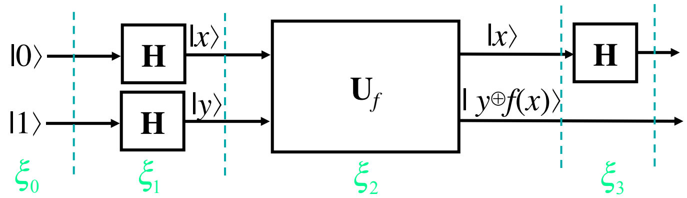
Deutsch's Problem



- When $f(0)\neq f(1)$:

$$|\xi_2\rangle = \begin{cases} \dfrac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T, & f(0)=0 \text{ and } f(1)=1 \\ -\dfrac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T, & f(0)=1 \text{ and } f(1)=0 \end{cases}$$

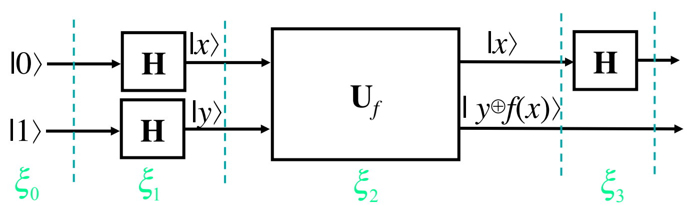
Deutsch's Problem



- Combining the case where $f(0)=f(1)$ and $f(0)\neq f(1)$:

$$|\xi_2\rangle=\begin{cases}\pm\left[\dfrac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=f(1)\\ \pm\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\dfrac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)\neq f(1)\end{cases}$$

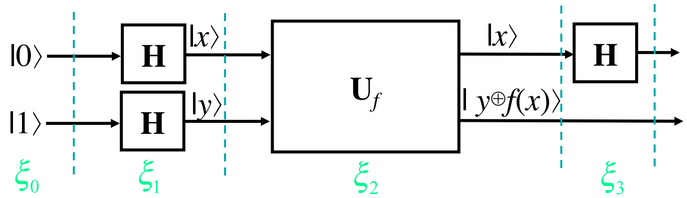
Deutsch's Problem



- Combining the case where $f(0)=f(1)$ and $f(0)\neq f(1)$:

$$|\xi_2\rangle=\begin{cases}\pm\dfrac{1}{2}\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^T, & f(0)=f(1)\\ \pm\dfrac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^T, & f(0)\neq f(1)\end{cases}$$

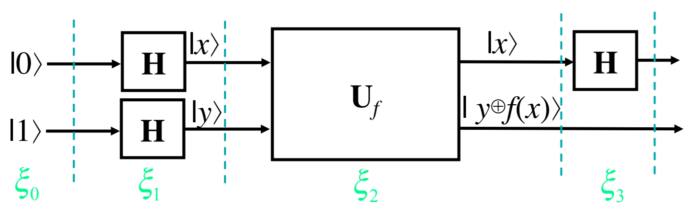
Deutsch's Problem



- Transfer Matrix for the Final Stage:

$$\mathbf{H}\otimes\mathbf{I}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}$$

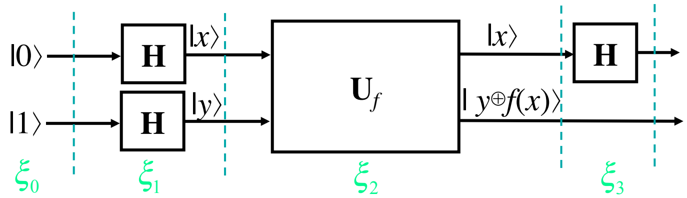
Deutsch's Problem



- When $f(0)=f(1)$:

$$|\xi_3\rangle=\pm\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix}=\pm\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\\ 0\\ 0\end{bmatrix}=\pm|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

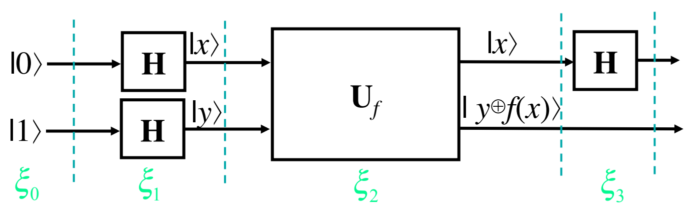
Deutsch's Problem



- When $f(0) \neq f(1)$:

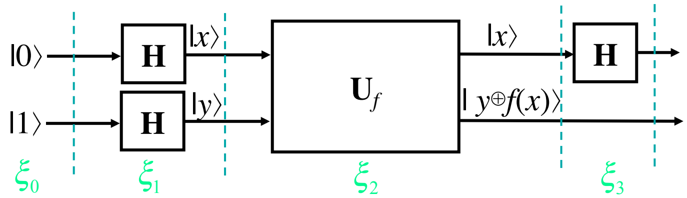
$$|\xi_3\rangle = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \pm|1\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Deutsch's Problem



- Measuring First Output Qubit of Circuit Yields Answer to Problem
- For $f(0)=f(1)$: $$|\xi_3\rangle = \pm|0\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \pm\frac{|00\rangle - |01\rangle}{\sqrt{2}}$$
- For $f(0) \neq f(1)$: $$|\xi_3\rangle = \pm|1\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \pm\frac{|10\rangle - |11\rangle}{\sqrt{2}}$$

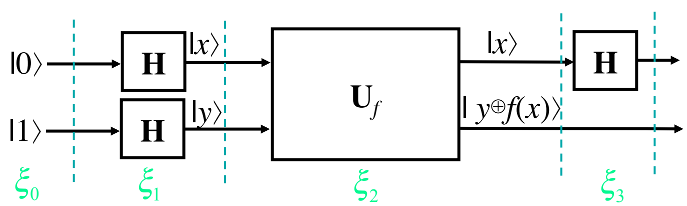
Deutsch's Problem



- For $f(0)=f(1)$, Measuring First Output Qubit is Equally likely to give one of the two basis vectors $|00\rangle$ or $|01\rangle$:

- For $f(0)\neq f(1)$, Measuring First Output Qubit is Equally likely to give one of the two basis vectors $|10\rangle$ or $|11\rangle$:

Deutsch's Problem



- Measuring First Output Qubit Yields the Answer
- First Output $= |0\rangle$ means $f(0)=f(1)$
- First Output $= |1\rangle$ means $f(0)\neq f(1)$
- Can Rewrite as:

$$|\xi_3\rangle = \pm\,|f(0)\oplus f(1)\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

Generalized Deutsch's Problem

- Original Deutsch Problem dealt with identification of one of four possible single bit functions:

$$\{0,1\} \to \{0,1\}$$

- This Problem is Generalized for Functions of the Form:

$$\{0,1\}^n \to \{0,1\}$$

- The Quantum Solution of this Generalized Problem is Called the Deutsch-Jozsa Algorithm

Deutsch-Jozsa Agorithm

- Functions of this form can be classified as balanced or unbalanced

$$\{0,1\}^n \to \{0,1\}$$

- Balanced functions are those that Output a "1" for half the input terms and a "0" for the other half
- Unbalanced Functions Output a Constant "1" or "0" Regardless of the input terms
- For a Function whose domain consists of n bitstrings, there are 2 Unbalanced Functions
- For a Function whose domain consists of n bitstrings, there are N Unbalanced Functions

Number of Balanced Functions

- For a Function whose domain consists of n bitstrings, there are N Unbalanced Functions
- N can be Computed using the binomial coefficient:

$$N = \binom{2^n}{2^{n-1}}$$

- This is a "special" coefficient known as the "central binomial coefficient"
- The central binomial coefficient is of the following form and obeys the identity: $\binom{2x}{x} = \frac{(2x)!}{(x!)^2}$

Number of Balanced Functions

- Using the property of the central binomial coefficient: $N = \binom{2^n}{2^{n-1}} = \frac{2^n!}{(n!)^2}$
- The sequence of central binomial coefficients is:

$$1,2,6,20,70,252,924,3432,12870,48620,...$$

- Some properties are the generating function:

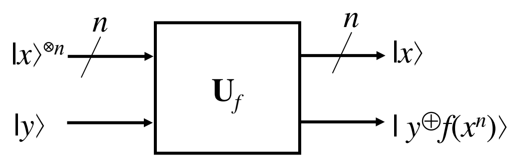
$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + ...$$

- Behavior in the limit (using Stirling's formula):

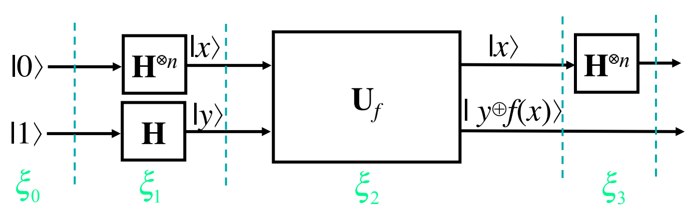
$$\lim_{n\to\infty} \binom{2^n}{2^{n-1}} = \frac{2^{n+1}}{\sqrt{\pi 2^{n-1}}}$$

Deutsch-Jozsa Agorithm

- $\mathbf{U}_f$ is Sometimes Called the "Oracle"
- The Oracle has a function embedded in it that is either balanced or unbalanced (constant)
- Note this does not include all possible functions!



Deutsch-Jozsa Agorithm



- This is NOT a particularly useful algorithm
- It is a good example to show how an exponential problem (in terms of Turing computation) can be accomplished in constant time with a quantum computer