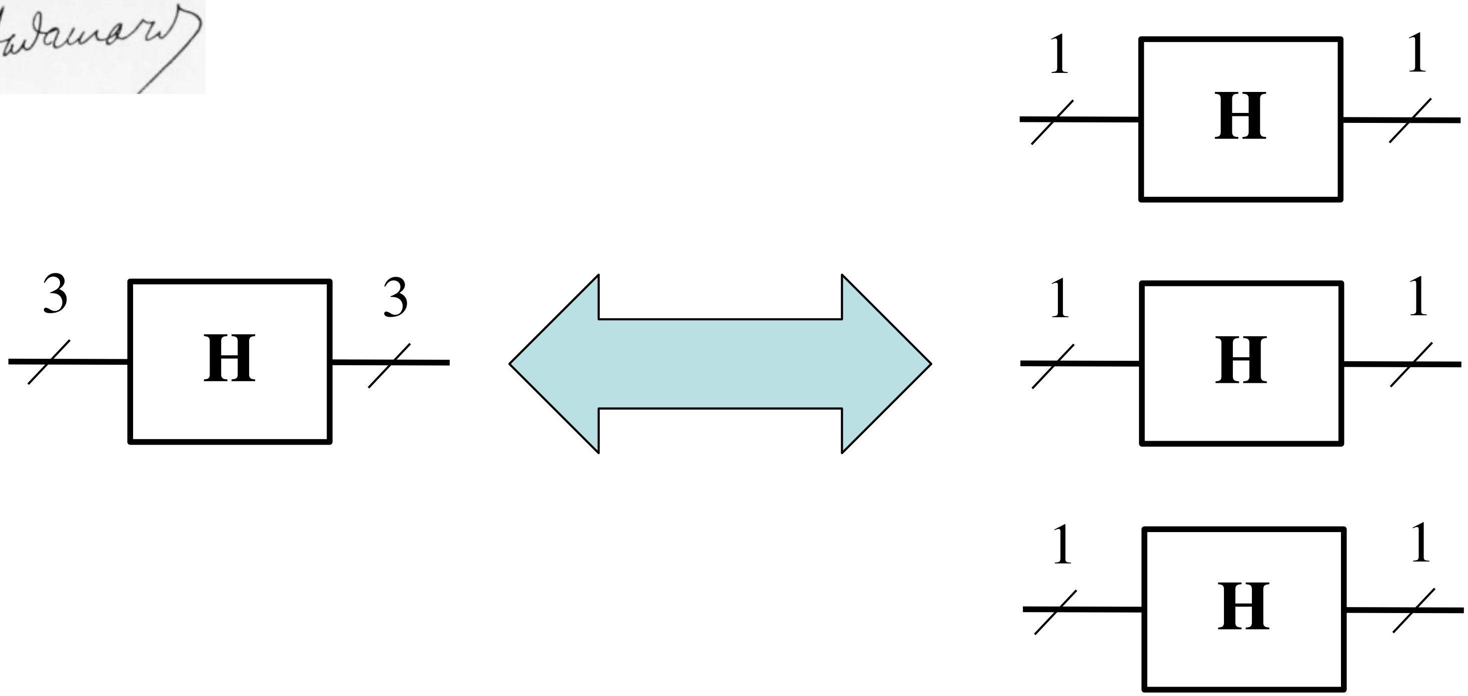
Hadamard Matrices/Operators

$$\mathbf{H}^{\otimes 3} = \frac{1}{\sqrt{2^3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$



Hadamard Matrices

- Square Matrices with Mutually Orthonormal Rows/Columns
- All Matrix Elements are Either $+1$ or -1
- In Signal Processing, Known as the "Walsh Transform"
- Walsh Transform is Fourier Transform with Square Waves (Walsh Functions) as Basis Functions
  - Fourier Transform on Two-Element Additive Group $\mathbb{Z}_2 : (\{-1,+1\},+_2)$
- Different Row Orderings Yield Variations of the Walsh Matrix

Hadamard Matrices

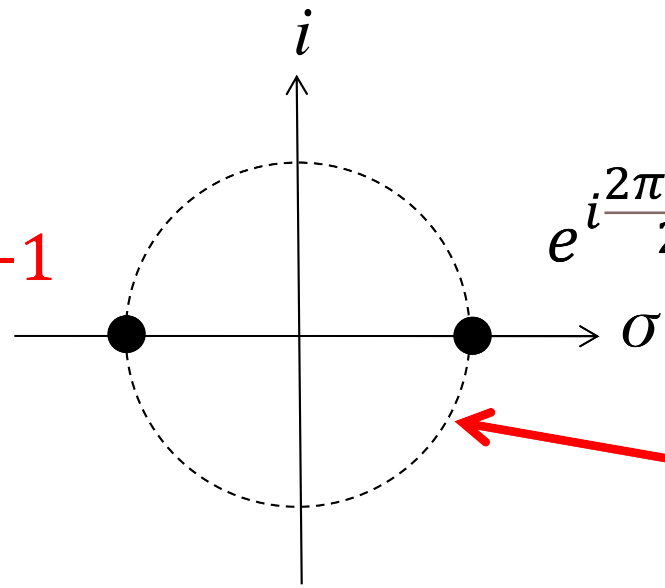
- Natural Row Ordering Defined by Outer/Tensor (Kronecker Product)
- Rademacher-Walsh Ordering Defined by XOR Operations among Adjacent Rows
- Transform can be Implemented Using $n\log n$ Operations ("Fast" Transform)
  - Can factor as sparse direct product factors
- Certain Forms can be Used Directly as Error Correcting Codes
- One Form is Known as the Reed-Muller Codes/Transform

Hadamard Matrix with Natural Ordering

+1 is Square Root of 1

-1 is Square Root of 1

$$e^{i\frac{2\pi(1)}{2}} = -1$$



$$e^{i\frac{2\pi(0)}{2}} = +1$$

Unit Circle in Complex Plane

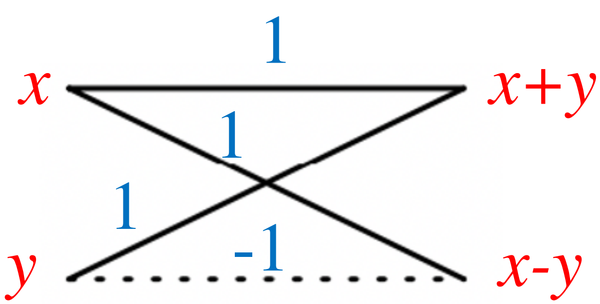
- This Form uses Square Roots of Unity Shown as Points on the Unit Circle in the Complex Plane
- Transform is a Discrete Fourier Transform over $GF(2)$
- Can Think of this as a Discrete Fourier Transform with Discretized Orthogonal Square Wave Functions as the Basis Set

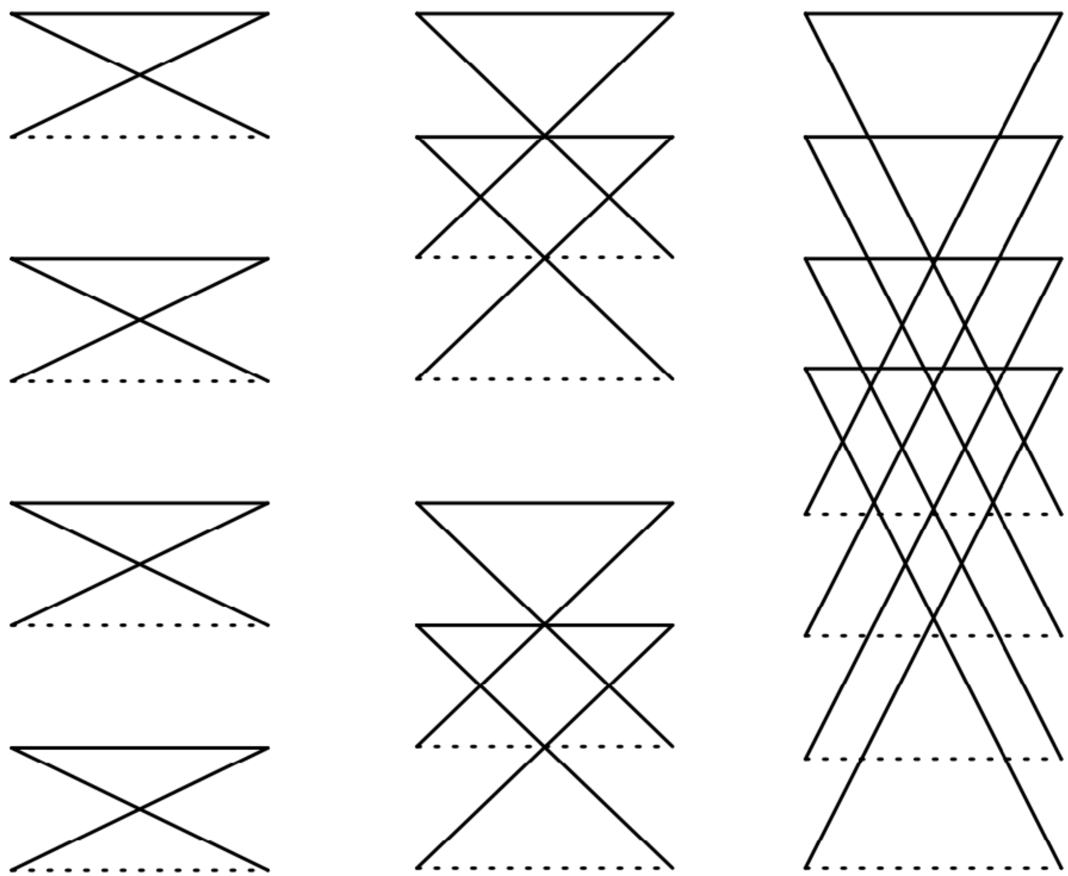
Fast Hadamard Transform

- So-called "fast" transforms and Butterfly Diagrams (Signal Flow Graphs)

$$\mathbf{H}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

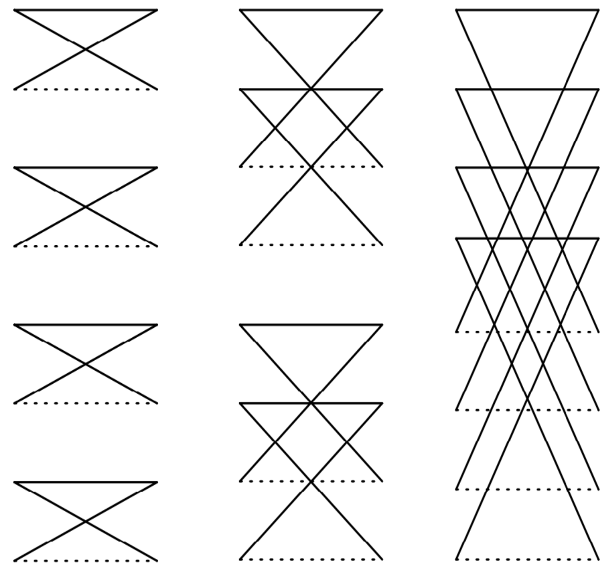
$$= \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$





Fast Hadamard Transform

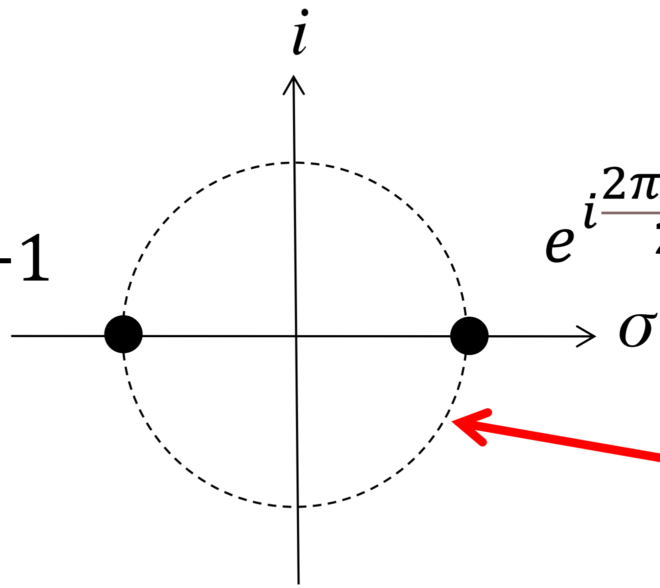
- So-called "fast" transforms due to Sparse Factors



$$\mathbf{H}^{\otimes 3} = \frac{1}{\sqrt{2^3}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Hadamard Matrix Rademacher-Walsh Ordering

$$e^{i\frac{2\pi(1)}{2}} = -1$$



$$e^{i\frac{2\pi(0)}{2}} = +1$$

$+1$ Maps to Integer 0

-1 Maps to Integer 1

Unit Circle in Complex Plane

- This Form uses Boolean Logic Values Instead of Mappings to the Unit Circle in the Complex Plane
- Transform Yield a Form of ESOP in Classical Logic

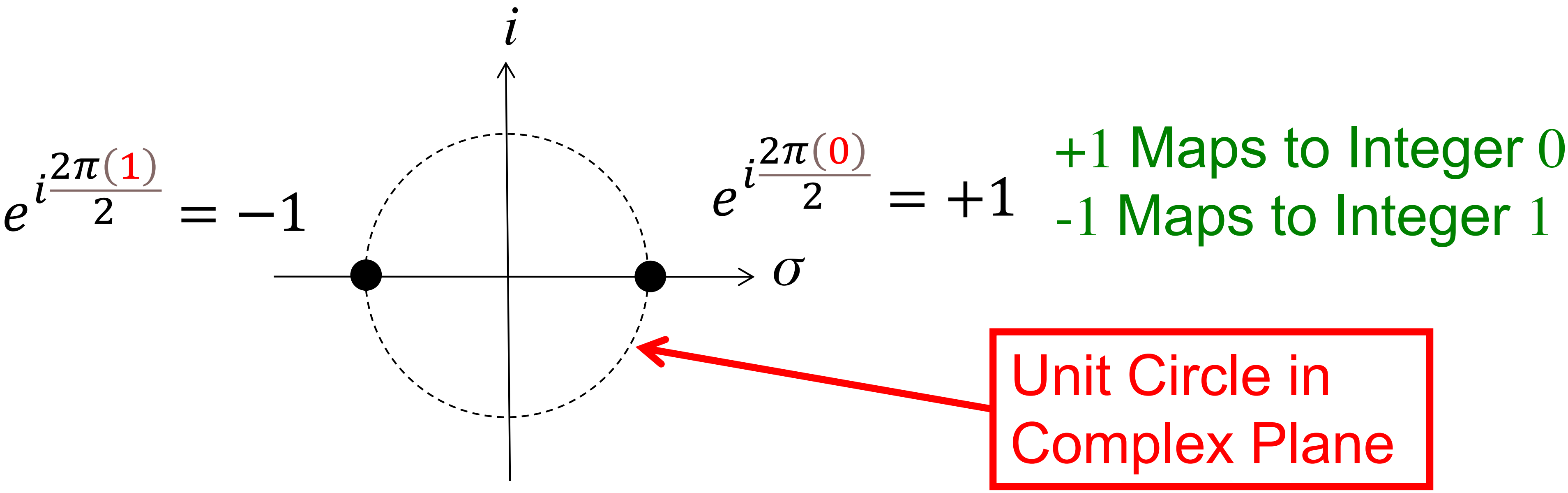
Rademacher-Walsh Transform

- Same as the Naturally-ordered Hadamard Transform with Rows/Columns Permuted
- Other Orderings Possible
- Referred to as "Walsh Transforms" in the Signal Processing Community
- Sometimes the Scale Factor $\left(1/\sqrt{2}\right)^n$ is Not Used in Signal Processing Applications

Map 0 to +1
Map 1 to -1

$$
\mathbf{H}_{RW} = \begin{array}{c} \left[\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{array}\right] \end{array}
\begin{array}{c}
0 \\ x \\ y \\ z \\ y \oplus z \\ x \oplus z \\ x \oplus y \\ x \oplus y \oplus z
\end{array}
$$

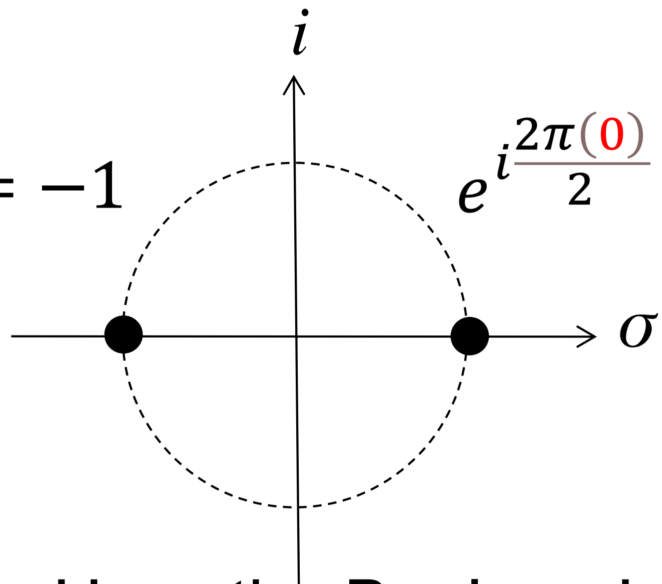
Reed-Muller Matrix with Natural Ordering



- This Form uses Boolean Logic Values Instead of Mappings to the Unit Circle in the Complex Plane
- Transform Yield a Form of ESOP in Classical Logic

Reed-Muller Form of Hadamard Matrix

$$e^{i\frac{2\pi(1)}{2}} = -1$$



$$e^{i\frac{2\pi(0)}{2}} = +1$$

+1 Maps to Boolean 0
-1 Maps to Boolean 1

Use $\mathbb{B}=\{0,1\}$ Instead of $\mathbb{Z}_2$

- This Form Uses the Boolean Logic Values Instead of Mappings to the Unit Circle in the Complex Plane
- Transform Yields a Form of ESOP in Classical Logic

$$\mathbf{R}^{\otimes 3} = \mathbf{R}_3 = \mathbf{R}_1 \otimes \mathbf{R}_1 \otimes \mathbf{R}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Reed-Muller Form of Hadamard Matrix

$$\mathbf{R}^{\otimes 3} = \mathbf{R}_3 = \mathbf{R}_1 \otimes \mathbf{R}_1 \otimes \mathbf{R}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{R}^{\otimes 3} = \otimes_{i=1}^{3} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

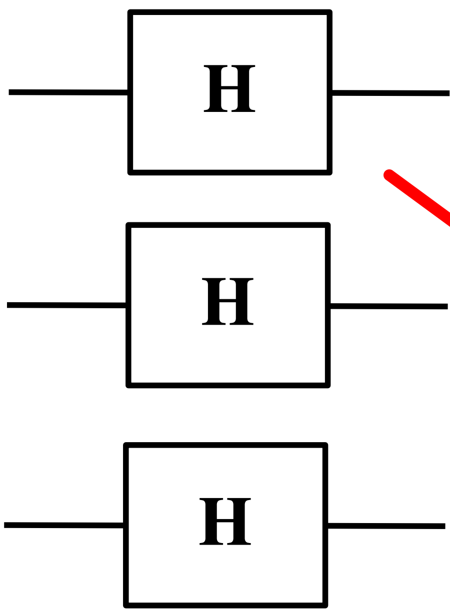
$$\mathbf{R}^{\otimes 3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For More Details of Classical Logic Synthesis of ESOPs: CSE 8387 Switching Theory Class

Naturally Ordered Hadamard

$$\mathbf{H}^{\otimes 3} = \mathbf{H}\otimes\mathbf{H}\otimes\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{H}^{\otimes 3} = \mathbf{H}\otimes\mathbf{H}\otimes\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2^2}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$



$$\mathbf{H}^{\otimes 3} = \frac{1}{\sqrt{2^3}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Hadamard & Superposition

$$\mathbf{H}^{\otimes 3}|000\rangle = \frac{1}{\sqrt{2^3}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{H}^{\otimes 3}|000\rangle = \frac{1}{\sqrt{2^3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^3}}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\mathbf{H}^{\otimes 3}|000\rangle = \frac{1}{\sqrt{2^3}}(|000\rangle + |001\rangle \cdots + |111\rangle)$$

Naturally Ordered Hadamard

- Alternative Notation (using symbolic logic)
  - Conjunctive Logic Operation (Binary AND function): $\wedge$

$$\mathbf{H} = \left[h_{ij}\right] = \frac{1}{\sqrt{2}}(-1)^{i \wedge j}$$

- i and j are row and column numbers
- Can Rewrite Hadamard Matrix as:

column numbers

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{array}{cc} 0 & 1 \end{array} & \\ \begin{bmatrix} (-1)^{0\wedge 0} & (-1)^{0\wedge 1} \\ (-1)^{1\wedge 0} & (-1)^{1\wedge 1} \end{bmatrix} & \begin{array}{c} 0 \\ 1 \end{array} \end{array}$$

row numbers

Naturally Ordered Hadamard

$$\mathbf{H}^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{matrix} \quad 0 & \qquad\qquad 1 \end{matrix} & \\ \begin{bmatrix} (-1)^{0\wedge 0} & (-1)^{0\wedge 1} \\ (-1)^{1\wedge 0} & (-1)^{1\wedge 1} \end{bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array} \otimes \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{matrix} \quad 0 & \qquad\qquad 1 \end{matrix} & \\ \begin{bmatrix} (-1)^{0\wedge 0} & (-1)^{0\wedge 1} \\ (-1)^{1\wedge 0} & (-1)^{1\wedge 1} \end{bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}$$

$$\mathbf{H}^{\otimes 2} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) \begin{array}{cc} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} & \\ \left[\begin{array}{cc|cc} (-1)^{0\wedge 0} \times (-1)^{0\wedge 0} & (-1)^{0\wedge 0} \times (-1)^{0\wedge 1} & (-1)^{0\wedge 1} \times (-1)^{0\wedge 0} & (-1)^{0\wedge 1} \times (-1)^{0\wedge 1} \\ (-1)^{0\wedge 0} \times (-1)^{1\wedge 0} & (-1)^{0\wedge 0} \times (-1)^{1\wedge 1} & (-1)^{0\wedge 1} \times (-1)^{1\wedge 0} & (-1)^{0\wedge 1} \times (-1)^{1\wedge 1} \\ \hline (-1)^{1\wedge 0} \times (-1)^{0\wedge 0} & (-1)^{1\wedge 0} \times (-1)^{0\wedge 1} & (-1)^{1\wedge 1} \times (-1)^{0\wedge 0} & (-1)^{1\wedge 1} \times (-1)^{0\wedge 1} \\ (-1)^{1\wedge 0} \times (-1)^{1\wedge 0} & (-1)^{1\wedge 0} \times (-1)^{1\wedge 1} & (-1)^{1\wedge 1} \times (-1)^{1\wedge 0} & (-1)^{1\wedge 1} \times (-1)^{1\wedge 1} \end{array}\right] & \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \end{array}$$

- Multiply $(-1)^x$ by $(-1)^y = (-1)^{x+y}$:

$$\mathbf{H}^{\otimes 2} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) \begin{array}{cc} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} & \\ \left[\begin{array}{cc|cc} (-1)^{0\wedge 0+0\wedge 0} & (-1)^{0\wedge 0+0\wedge 1} & (-1)^{0\wedge 1+0\wedge 0} & (-1)^{0\wedge 1+0\wedge 1} \\ (-1)^{0\wedge 0+1\wedge 0} & (-1)^{0\wedge 0+1\wedge 1} & (-1)^{0\wedge 1+1\wedge 0} & (-1)^{0\wedge 1+1\wedge 1} \\ \hline (-1)^{1\wedge 0+0\wedge 0} & (-1)^{1\wedge 0+0\wedge 1} & (-1)^{1\wedge 1+0\wedge 0} & (-1)^{1\wedge 1+0\wedge 1} \\ (-1)^{1\wedge 0+1\wedge 0} & (-1)^{1\wedge 0+1\wedge 1} & (-1)^{1\wedge 1+1\wedge 0} & (-1)^{1\wedge 1+1\wedge 1} \end{array}\right] & \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \end{array}$$

$$\mathbf{H}^{\otimes 2} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) \begin{array}{cc} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} & \\ \left[\begin{array}{cc|cc} (-1)^{0} & (-1)^{0} & (-1)^{0} & (-1)^{0} \\ (-1)^{0} & (-1)^{1} & (-1)^{0} & (-1)^{1} \\ \hline (-1)^{0} & (-1)^{0} & (-1)^{1} & (-1)^{1} \\ (-1)^{0} & (-1)^{1} & (-1)^{1} & (-1)^{2} \end{array}\right] & \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \end{array} = \frac{1}{\sqrt{2^2}} \left[\begin{array}{rr|rr} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

Naturally Ordered Hadamard

- Multiplying $(-1)^x$ by $(-1)^y$ = $(-1)^{x+y}$: Yields $(-1)^{(2n)}$ or $(-1\ )^{(2n)+1}$
- Where n is an Integer $n\in\mathbb{Z}$, $\mathbb{Z}=\{0, 1, 2, ...\}$
- Exponentiating -1 to a non-negative Integer Results in:

$$(-1)^{2n} = +1 \qquad\qquad (-1)^{2n+1} = -1$$

- Therefore,

$$(-1)^{(i\wedge j)+(k\wedge m)} = (-1)^{2n} = +1$$

$$(-1)^{(i\wedge j)+(k\wedge m)} = (-1)^{2n+1} = -1$$

- We Only Need to Consider if Exponent is Even or Odd
  - When: $(i \wedge j) + (k \wedge m) = 2n \longrightarrow (-1)^{(i\wedge j)+(k\wedge m)} = +1$
  - When: $(i \wedge j) + (k \wedge m) = 2n + 1 \longrightarrow (-1)^{(i\wedge j)+(k\wedge m)} = -1$
- This is Modulo-2 Addition, can Replace with Exclusive-OR

Naturally Ordered Hadamard

- We Only Need to Consider if Exponent is Even or Odd
  - When: $(i \wedge j) + (k \wedge m) = 2n \longrightarrow (-1)^{(i \wedge j)+(k \wedge m)} = +1$
  - When: $(i \wedge j) + (k \wedge m) = 2n + 1 \longrightarrow (-1)^{(i \wedge j)+(k \wedge m)} = -1$
- This is Modulo-2 Addition, can Replace with Exclusive-OR
  - When: $(i \wedge j) + (k \wedge m) = 2n \longrightarrow (i \wedge j) \oplus (k \wedge m) = 0$
  - When: $(i \wedge j) + (k \wedge m) = 2n + 1 \longrightarrow (i \wedge j) \oplus (k \wedge m) = 1$
- Can Express $\mathbf{H}^{\otimes 2}$ Hadamard Matrix as:

$$\mathbf{H}^{\otimes 2} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} (-1)^{0\wedge 0 \oplus 0 \wedge 0} & (-1)^{0\wedge 0 \oplus 0 \wedge 1} & (-1)^{0\wedge 1 \oplus 0 \wedge 0} & (-1)^{0\wedge 1 \oplus 0 \wedge 1} \\ (-1)^{0\wedge 0 \oplus 1 \wedge 0} & (-1)^{0\wedge 0 \oplus 1 \wedge 1} & (-1)^{0\wedge 1 \oplus 1 \wedge 0} & (-1)^{0\wedge 1 \oplus 1 \wedge 1} \\ (-1)^{1\wedge 0 \oplus 0 \wedge 0} & (-1)^{1\wedge 0 \oplus 0 \wedge 1} & (-1)^{1\wedge 1 \oplus 0 \wedge 0} & (-1)^{1\wedge 1 \oplus 0 \wedge 1} \\ (-1)^{1\wedge 0 \oplus 1 \wedge 0} & (-1)^{1\wedge 0 \oplus 1 \wedge 1} & (-1)^{1\wedge 1 \oplus 1 \wedge 0} & (-1)^{1\wedge 1 \oplus 1 \wedge 1} \end{bmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$\mathbf{H}^{\otimes 2} = \frac{1}{\sqrt{2^2}} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} (-1)^0 & (-1)^0 & (-1)^0 & (-1)^0 \\ (-1)^0 & (-1)^1 & (-1)^0 & (-1)^1 \\ (-1)^0 & (-1)^0 & (-1)^1 & (-1)^1 \\ (-1)^0 & (-1)^1 & (-1)^1 & (-1)^0 \end{bmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} = \frac{1}{\sqrt{2^2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Notation

- Using this Form for Naturally Ordered Hadamard
  - Following Function Notation is Helpful
  - DO NOT Confuse with BraKet EXPECTED VALUE!!!!
  - The COMMA is Important (this is NOT $\langle \mathbf{A} \rangle$ or $\langle \Psi | \mathbf{A} | \Psi \rangle$)

$$\langle \, , \rangle : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

  - Inner Product Function over Strings, $\mathbb{B}^{2n} = \{0,1\}^{2n} \longrightarrow \mathbb{B}$

EXAMPLE: Two n-bit Strings

$$\mathbf{x} = x_{n-1} x_{n-1} \cdots x_2 x_1 x_0 \qquad \mathbf{y} = y_{n-1} y_{n-1} \cdots y_2 y_1 y_0$$

  - Note, We Use Bit-wise Exclusive-OR (a string):

$$\mathbf{x} \oplus \mathbf{y} = (x_{n-1} \oplus y_{n-1}), (x_{n-1} \oplus y_{n-1}), \cdots, (x_1 \oplus y_1), (x_0 \oplus y_0)$$

  - The "Inner Product" Notation is a Single Value:

$$\langle \mathbf{x}, \mathbf{y} \rangle = (x_{n-1} \wedge y_{n-1}) \oplus (x_{n-1} \wedge y_{n-1}) \oplus \cdots \oplus (x_1 \wedge y_1) \oplus (x_0 \wedge y_0)$$

Bit-String Inner Product Properties

$$\langle\ ,\ \rangle : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

$$\langle \mathbf{x}, \mathbf{0} \rangle = \mathbf{0} \qquad \langle \mathbf{0}, \mathbf{y} \rangle = \mathbf{0}$$

$$\langle \mathbf{x}, \mathbf{1} \rangle = x_{n-1} \oplus x_{n-2} \oplus \cdots \oplus x_1 \oplus x_0 \qquad \langle \mathbf{1}, \mathbf{y} \rangle = y_{n-1} \oplus y_{n-2} \oplus \cdots \oplus y_1 \oplus y_0$$

$$\langle \mathbf{x} \oplus \bar{\mathbf{x}}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \oplus \langle \bar{\mathbf{x}}, \mathbf{y} \rangle \qquad \langle \mathbf{x}, \mathbf{y} \oplus \bar{\mathbf{y}} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \oplus \langle \mathbf{x}, \bar{\mathbf{y}} \rangle$$

$$\langle \mathbf{x} \oplus \bar{\mathbf{x}}, \mathbf{y} \rangle = \langle \mathbf{1}, \mathbf{y} \rangle \qquad \langle \mathbf{x}, \mathbf{y} \oplus \bar{\mathbf{y}} \rangle = \langle \mathbf{x}, \mathbf{1} \rangle$$

$$\langle \mathbf{0} \wedge \mathbf{x}, \mathbf{y} \rangle = \langle 0^n, \mathbf{y} \rangle = 0 \qquad \langle \mathbf{x}, \mathbf{0} \wedge \mathbf{y} \rangle = \langle \mathbf{x}, 0^n \rangle = 0$$

$$\langle \mathbf{1} \wedge \mathbf{x}, \mathbf{y} \rangle = \langle 1^n \wedge \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \qquad \langle \mathbf{x}, \mathbf{1} \wedge \mathbf{y} \rangle = \langle \mathbf{x}, 1^n \wedge \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$$

Hadamard with New Notation

$$\mathbf{H}^{\otimes 2} = \frac{1}{\sqrt{2^2}} \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{bmatrix} (-1)^0 & (-1)^0 & (-1)^0 & (-1)^0 \\ (-1)^0 & (-1)^1 & (-1)^0 & (-1)^1 \\ (-1)^0 & (-1)^0 & (-1)^1 & (-1)^1 \\ (-1)^0 & (-1)^1 & (-1)^1 & (-1)^0 \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array} = \frac{1}{\sqrt{2^2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{H}^{\otimes 2} = \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{bmatrix} (-1)^{\langle 00,00 \rangle} & (-1)^{\langle 00,01 \rangle} & (-1)^{\langle 01,00 \rangle} & (-1)^{\langle 01,01 \rangle} \\ (-1)^{\langle 00,10 \rangle} & (-1)^{\langle 00,11 \rangle} & (-1)^{\langle 01,10 \rangle} & (-1)^{\langle 01,11 \rangle} \\ (-1)^{\langle 10,00 \rangle} & (-1)^{\langle 10,01 \rangle} & (-1)^{\langle 11,00 \rangle} & (-1)^{\langle 11,01 \rangle} \\ (-1)^{\langle 10,10 \rangle} & (-1)^{\langle 10,11 \rangle} & (-1)^{\langle 11,10 \rangle} & (-1)^{\langle 11,11 \rangle} \end{bmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

Naturally Ordered Hadamard

- Now Can Write General Formula for:

$$\mathbf{H}^{\otimes n}(\mathbf{i},\mathbf{j}) = \frac{1}{\sqrt{2^n}}(-1)^{\langle \mathbf{i},\mathbf{j}\rangle}$$

- $\mathbf{i}$ and $\mathbf{j}$ are row and column numbers written as binary strings
- Quantum State Vector (example):

$$|\mathbf{0}\rangle = |000\cdots00\rangle = \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} & \begin{array}{c} \color{red}{000\cdots00} \\ \color{red}{000\cdots01} \\ \color{red}{000\cdots10} \\ \vdots \\ \color{red}{111\cdots10} \\ \color{red}{111\cdots11} \end{array} \end{array}$$

Red strings are row numbers written as binary strings – NOT part of equation

Naturally Ordered Hadamard

- Multiplying a Quantum State Vector by $\mathbf{H}^{\otimes n}$

Leftmost Column of Hadamard Matrix

$$\mathbf{H}^{\otimes n}|\mathbf{0}\rangle = \mathbf{H}^{\otimes n}[-,\mathbf{0}] = \frac{1}{\sqrt{2^n}}\begin{array}{cc} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} & \begin{array}{c} 000\cdots00 \\ 000\cdots01 \\ 000\cdots10 \\ \vdots \\ 111\cdots10 \\ 111\cdots11 \end{array} \end{array} = \frac{1}{\sqrt{2^n}}\sum_{\mathbf{x}\in\{0,1\}^n}|\mathbf{x}\rangle$$

- For Arbitrary Quantum State $|\mathbf{y}\rangle$

$$\mathbf{H}^{\otimes n}|\mathbf{y}\rangle = \mathbf{H}^{\otimes n}[-,\mathbf{y}] = \frac{1}{\sqrt{2^n}}\sum_{\mathbf{x}\in\{0,1\}^n}(-1)^{\langle\mathbf{x},\mathbf{y}\rangle}|\mathbf{x}\rangle$$

- Denotes "don't care" or All Possible Rows from 0 to n-1