Chrestenson Spectrum Computation Using Cayley Color Graphs*

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Abstract

A method based on eigenvalue computations is formulated for computing the Chrestenson spectrum of a discrete p-valued function. This technique is developed by considering an extension to the same approach of computation of the Walsh spectrum for a two-valued function and is then generalized to the p-valued case. Algebraic groups are formulated that correspond to Cayley color graphs based on the function of interest whose adjacency matrices have spectra equivalent to the Walsh or Chrestenson spectrum of the function under consideration. Because the transformation matrix is not used in any of these computations, the method provides an alternative approach for spectral computations.

1 Introduction

In [1] a method for computing the Walsh spectrum of a completely specified Boolean function was presented that is based on the formation of an algebraic group dependent on the Boolean function yielding a corresponding Cayley graph [6, 7, 25]. It is shown in [1] that the spectrum of the Cayley graph is in fact equivalent to the Walsh spectrum of the Boolean function. Some extensions and applications of this approach are provided in [2, 3, 24].

A generalization of the set of orthogonal Walsh functions over the binary field leads to the set of Chrestenson functions [8]. These functions may be used as basis functions for a discrete transform of a p-valued function where $p > 2$. Several researchers have investigated the computation and application of the Chrestenson spectrum for p-valued discrete functions and in particular, the ternary case [18, 19, 22]. Applications that have been considered in the past are synthesis [4, 11, 14], decomposition [20, 23] and testing [10, 17] among others.

In the following it is proven that the spectrum of a Cayley color graph formulated from an algebraic group similar to that used in [1] yields the Walsh spectrum in S-encoding for a fully specified binary-valued function. These results are then generalized and it is shown that the Chrestenson spectrum of a fully specified p-valued function may likewise be computed as the spectrum of a Cayley color graph. A specific example is given for the case of a ternary-valued function.

The remainder of this paper is organized as follows. Section 2 will provide a brief overview of the computation of the Chrestenson spectrum of a ternary-valued function and the pertinent results from [1]. Section 3 will give extensions of the results of [1] and show how these extensions are applicable to the p-valued case. In Section 4, an example computation is given showing how the method can be used. Conclusions are provided in Section 5.

2 Preliminaries

The computation of the Chrestenson spectrum of a fully specified ternary-valued discrete function using the mathematically defined methods such as those described in [16] is briefly reviewed followed by an overview of the pertinent aspects of [1]. The need for a review of the concepts in [1, 16] is motivated by the fact that extensions of the method as presented in Section 3 are used for the p-valued case of discrete functions.

2.1 Chrestenson Spectrum of Ternary Function

By representing all discrete function values as roots of unity in the complex plane as shown in Figure 1 for ternary valued functions, a direct mapping to spectral representations results. In this mapping, the ternary logic values of $\{0, 1, 2\}$ are mapped to values in the complex plane $\{a_0, a_1, a_2\}$ respectively where $a_0 = e^{j0} = 1$, $a_1 = e^{j\frac{2\cdot\pi}{3}}$ and $a_2 = e^{j\frac{4\cdot\pi}{3}}$.

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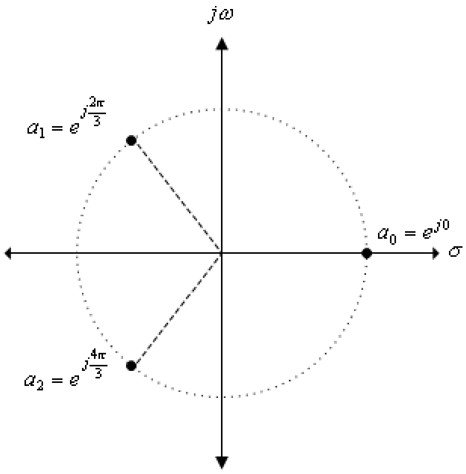

Figure 1. Complex Plane with Encoded Ternary Function Values

To compute the Chrestenson spectrum of a ternary-valued function, all logic-0 values are mapped to a_0, logic-1 values to $a_1^* = a_2$ and logic-2 values to $a_2^* = a_1$. The superscript $*$ denotes the complex conjugate of the values that correspond to the original function values. This is necessary for the inverse transform to yield the proper values. The resulting vector of complex values is then linearly transformed as a matrix-vector product. The Chrestenson transformation matrix may be formulated in natural order using the Kronecker product definition [12] as shown in Equations 1 and 2.

$$\mathbf{C^0} = [\mathbf{1}] \tag{1}$$

$$\mathbf{C^n} = \bigotimes_{\mathbf{i=1}}^{\mathbf{n}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a_1 & a_2 \\ 1 & a_2 & a_1 \end{bmatrix} \tag{2}$$

As an example, the function f defined by the truth table shown in Table 1 is transformed using the transformation matrix in Equation 3. In Table 1 the fourth column, f_{ENC}, contains the function values after encoding into the complex conjugates of the roots of unity shown in Figure 1.

Table 1. Truth Table of Example Ternary Function

x_1	x_2	f	f_{ENC}
0	0	0	a_0
0	1	2	a_1
0	2	0	a_0
1	0	1	a_2
1	1	0	a_0
1	2	2	a_1
2	0	2	a_1
2	1	1	a_2
2	2	0	a_0

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & 1 & a_1 & a_2 & 1 & a_1 & a_2 \\ 1 & a_2 & a_1 & 1 & a_2 & a_1 & 1 & a_2 & a_1 \\ 1 & 1 & 1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 \\ 1 & a_1 & a_2 & a_1 & a_2 & 1 & a_2 & 1 & a_1 \\ 1 & a_2 & a_1 & a_1 & 1 & a_2 & a_2 & a_1 & 1 \\ 1 & 1 & 1 & a_2 & a_2 & a_2 & a_1 & a_1 & a_1 \\ 1 & a_1 & a_2 & a_2 & 1 & a_1 & a_1 & a_2 & 1 \\ 1 & a_2 & a_1 & a_2 & a_1 & 1 & a_1 & 1 & a_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_0 \\ a_2 \\ a_0 \\ a_1 \\ a_1 \\ a_2 \\ a_0 \end{bmatrix} = \begin{bmatrix} 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 2a_0 + 4a_1 + 3a_2 \\ 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 8a_0 + a_1 \\ 4a_0 + 3a_1 + 2a_2 \\ 3a_0 + 2a_1 + 4a_2 \\ 2a_0 + 4a_1 + 3a_2 \end{bmatrix} \tag{3}$$

2.2 Eigenvalue Technique - Binary Case

In [1] a group is formulated and described by $(M, \oplus)$ where $M = \{m_i\}$ is the set of all possible 2^n minterms for a Boolean function f and $\oplus$ is the exclusive-OR operation applied bit by bit over a pair (m_i, m_j). This group can be used with the function of interest f to form a Cayley graph. The $2^n \times 2^n$ adjacency matrix $\mathbf{A} = [\mathbf{a_{ij}}]$ of the Cayley graph is formed using the relationship given in Equation 4.

$$a_{ij} = f(m_i \oplus m_j) \tag{4}$$

Since Equation 4 evaluates to the Boolean constants 0 or 1 the resulting undirected graph contains non-weighted edges and yields a Cayley graph. The spectrum of the graph, defined as the set of eigenvalues of $\mathbf{A}$ [5, 9], is also the R-encoded Walsh spectrum. R-encoding refers to mapping function values of Boolean 0 to the integer 0 and Boolean 1 to the integer +1.

In [24] extensions of the method presented in [1] are given without proof. In particular it is shown that the Walsh spectrum of the complement of f, $\overline{f}$, results when the group $(M, \equiv)$ is used where $\equiv$ is the bitwise equivalence operator (the exclusive-NOR) and that the S-encoded spectrum results when the elements of $\mathbf{A}$ are mapped to the integer values +1 and -1. The mapping of the $\mathbf{A}$ elements to the two roots of unity $e^{j0} = +1$ and $e^{j\pi} = -1$ result in a Cayley color graph. The Cayley color graph is a completely connected graph with edge weights (i.e. colors) corresponding to +1 for a Boolean 0 constant and -1 for a Boolean constant 1.

3 Extensions Using Cayley Color Graphs

In contrast to R-encoding as was used in [1], it is also possible to generate the Walsh coefficients of a two-valued function as a computation of the eigenvalues of an S-encoded adjacency matrix as described in [24]. In the following subsection a proof of this technique is given and used as motivation for the extension of the technique to fully specified p-valued discrete functions and their corresponding Chrestenson spectra.

3.1 Extensions to the Two-valued Method

As described in [15, 21] and numerous other sources, conversion from R to S encoded spectra can be accomplished as given in Equations 5 and 6.

$$s_0 = 2^n - 2r_0 \tag{5}$$

$$s_i = -2r_i \tag{6}$$

R-encoding represents a mapping of function range values such that the Boolean constant 0 is mapped to the integer 0 and the Boolean constant 1 is mapped to the integer +1. S-encoding represents the mapping of the binary Boolean constants to the complex plane as two roots of unity. This mapping is illustrated in Figure 2 for two-valued functions.

Lemma 1 *Let* $\mathbf{J}$ *denote the* $k \times k$ *matrix of all 1's. Then the eigenvalues of* $\mathbf{J}$ *are* k *(with multiplicity one) and 0 (with multiplicity k-1).*

Proof: Since all rows are equal and nonzero, rank(J)=1. Since a $k \times k$ matrix of rank$(k-h)$ has at least h eigenvalues equal to 0, it is concluded that $\mathbf{J}$ has at least $k-1$ eigenvalues equal to 0. Since trace$(\mathbf{J}) = \mathbf{k}$ and the trace is the sum of the eigenvalues, it follows that the remaining eigenvalue of $\mathbf{J}$ is equal to k. □

Lemma 2 *The all 1's* $k \times k$ *matrix* $\mathbf{J}$ *and the adjacency matrix* $\mathbf{A}$ *defined by Equation 4 are simultaneously diagonalizable.*

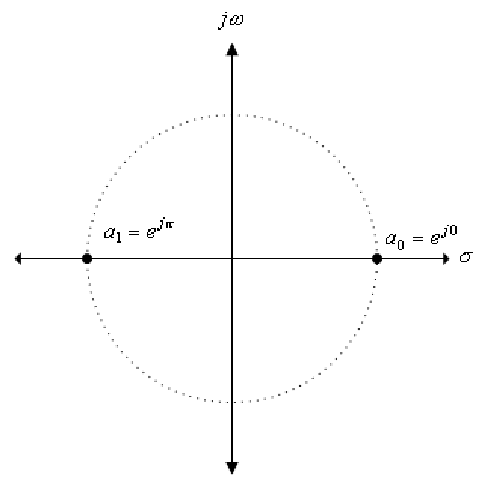

Figure 2. Complex Plane with Encoded Binary Function Values

Proof: From [13] it is proven that $\mathbf{J}$ and $\mathbf{A}$ are simultaneously diagonalizable if, and only if, they commute under matrix multiplication. Due to the definition of $\mathbf{A}$ in Equation 4, it is seen that $\mathbf{A} = \mathbf{A^T}$. Therefore, $\mathbf{AJ} = \mathbf{JA^T} = \mathbf{JA}$ thus $\mathbf{A}$ and $\mathbf{J}$ are simultaneously diagonalizable. □

Theorem 1 *The* S*-encoded Walsh coefficients of a two-valued function are given as the set of eigenvalues of the adjacency matrix* $\mathbf{A}$ *defined by Equation 4 with all 0-valued components replaced by +1 and all 1-valued components replaced by -1.*

Proof: The $2^n \times 2^n$ $\mathbf{A}$ matrix may be represented as an S-encoded version $\mathbf{B}$ as given by the relationship in Equation 7 where the $2^n \times 2^n$ $\mathbf{J}$ matrix is one with all elements equal to +1.

$$\mathbf{B} = \mathbf{J} - \mathbf{2A} \tag{7}$$

From Lemma 2, $\mathbf{J}$ and $\mathbf{A}$ are simultaneously diagonalizable and a nonsingular similarity matrix $\mathbf{S_m}$ exists that diagonalizes each. The diagonal matrices $\mathbf{D_J}$ and $\mathbf{D_A}$ corresponding to the similarity transform matrix $\mathbf{S_M}$ are given in Equations 8 and 9.

$$\mathbf{D_J} = \mathbf{S_m^{-1} J S_m} \tag{8}$$

$$\mathbf{D_A} = \mathbf{S_m^{-1} A S_m} \tag{9}$$

From Equations 8 and 9, it is observed that another diagonal matrix, $\mathbf{D_B}$ results as follows.

$\mathbf{D_B} = \mathbf{D_J} - \mathbf{2D_A} = \mathbf{S_m^{-1}(J - 2A)S_m} = \mathbf{S_m^{-1}JS_m} - \mathbf{2S_m^{-1}AS_m}$

The eigenvalues of $\mathbf{B}$ and $\mathbf{D_B}$ are the same due to the existence of the similarity transform $\mathbf{S_m}$. Thus, the eigenvalues of $\mathbf{B} = \mathbf{J} - \mathbf{2A}$ are of the form $\lambda_i - 2\mu_j$ where $\{\lambda_i\}$ and $\{\mu_i\}$ are the eigenvalues of $\mathbf{J}$ and $\mathbf{A}$ respectively. From Lemma 1 it is seen that the eigenvalues of $\mathbf{J}$ are $\lambda_0 = 2^n$ with all remaining $\lambda_i = 0$ for all $i \neq 0$.

Since $e = (1, 1, \ldots, 1)^T$ is an eigenvector of $\mathbf{J}$ corresponding to the eigenvalue $\lambda_0 = 2^n$ and $\mu_0 = \#1's$ in a row of $\mathbf{A}$ is an eigenvalue of $\mathbf{A}$, one eigenvalue of $\mathbf{B} = \mathbf{J} - \mathbf{2A}$ is $2^n - 2\mu_0$ which is of the form of s_0 as given in Equation 5. All other eigenvalues must be of the form $0 - 2\mu_j$ where $j \neq 0$ yielding the remaining Walsh spectral coefficients, s_i, as given in Equation 6. $\square$

3.2 Cayley Color Graph Spectrum for p-valued Functions

To extend the Cayley color graph spectrum computation method for Chrestenson spectrum calculations for fully specified p-valued functions, it is necessary to formulate an appropriate algebraic group. The $\oplus$ operator used for the case of two-valued functions can be viewed as an arithmetic operation in $GF(2)$. With this point of view $\oplus$ can be considered as addition or subtraction modulo-2 over the set of elements in M. While the addition operation modulo-p does indeed result in a Cayley color graph [25], the spectrum of the graph does not result in the corresponding Chrestenson spectrum for the function of interest. This is easy to see since the diagonal of the resulting adjacency matrix $\mathbf{A}$ does not contain same valued elements as is the case for two-valued functions; however, if the operator $\ominus_p$ is used where $\ominus_p$ represents the difference modulo-p, this characteristic is preserved. Hence, as is analogous to the technique for two-valued functions, a group $(M, \ominus_p)$ may be used and the adjacency matrix for a discrete p-valued function may be formed with each component defined by the relationship in Equation 10 where each a_{ij} is colored by the mapping defined above.

$$a_{ij} = f(m_i \ominus_p m_j) \tag{10}$$

4 Example Calculation

As an example of the extension of the Cayley color graph method for computing the Chrestenson spectrum of a fully specified p-valued function, consider the ternary-valued function (i.e. $p = 3$) defined by the truth table given in Table 1. Using the group and corresponding $\mathbf{A}$ matrix defined in Equation 10 the adjacency matrix given in Equation 11 results. Note that the entries of the adjacency matrix are in fact the complex conjugates of the function values when mapped to the roots of unity as shown in Figure 1. If complex conjugates are not used in the formation of $\mathbf{A}$, then the resulting eigenvalues of $\mathbf{A}$ are the complex conjugates of the spectrum [13].

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & a_1 & a_1 & 1 & a_2 & a_2 & a_1 & 1 \\ a_1 & 1 & 1 & a_2 & a_1 & 1 & 1 & a_2 & a_1 \\ 1 & a_1 & 1 & 1 & a_2 & a_1 & a_1 & 1 & a_2 \\ a_2 & a_1 & 1 & 1 & 1 & a_1 & a_1 & 1 & a_2 \\ 1 & a_2 & a_1 & a_1 & 1 & 1 & a_2 & a_1 & 1 \\ a_1 & 1 & a_2 & 1 & a_1 & 1 & 1 & a_2 & a_1 \\ a_1 & 1 & a_2 & a_2 & a_1 & 1 & 1 & 1 & a_1 \\ a_2 & a_1 & 1 & 1 & a_2 & a_1 & a_1 & 1 & 1 \\ 1 & a_2 & a_1 & a_1 & 1 & a_2 & 1 & a_1 & 1 \end{bmatrix} \tag{11}$$

The adjacency matrix given in Equation 11 has several structural properties typical of Cayley color graphs. $\mathbf{A}$ is a matrix with all values along the diagonal equivalent to the value of the function at the all-zero minterm $m_0 = 00\ldots0$. Also, the first column of $\mathbf{A}$ corresponds to the complex conjugate values of the function when mapped to the complex plane. When $\mathbf{A}$ is considered in terms of all $p \times p$ submatrix blocks (in this case $p = 3$), it is seen that $\mathbf{A}$ is a block circulant matrix with each submatrix being circulant. It is also noted that the matrix $\mathbf{A^T}$ can likewise be used to compute the Chrestenson spectrum since the eigenvalues of $\mathbf{A}$ are equivalent to those of $\mathbf{A^T}$ [13]. Obtaining $\mathbf{A}$ or $\mathbf{A^T}$ corresponds to taking the difference modulo-p of minterm values labeling the rows of $\mathbf{A}$ with those of the column or vice versa.

Computing the eigenvalues of $\mathbf{A}$ in Equation 11 results in the set of values $\lambda_i = (4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 2a_0 + 4a_1 + 3a_2, 4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 8a_0 + a_1, 4a_0 + 3a_1 + 2a_2, 3a_0 + 2a_1 + 4a_2, 2a_0 + 4a_1 + 3a_2)$ which are also the Chrestenson spectral coefficients as can be verified by comparison to Equation 3 where the spectrum is computed according to the definition.

5 Conclusions

The computation of the Chrestenson spectrum of a p-valued discrete function is shown to be accomplished by computing the spectrum of an adjacency matrix representing a Cayley color graph. The matrix is formed as a collection of column (or row) vectors each of which is a permutation of the ternary function truth vector when encoded in the complex plane. These results are of theoretical interest since they illustrate the relationship between the spectrum of a multi-valued discrete function and the eigenvalue computation problem.

Further work will focus on the use of these results to formulate efficient algorithms for the computation of Chrestenson spectra. Because the adjacency matrices are block circulant an implicit algorithm may be formed using decision diagrams to represent the function of interest. Exploiting the block circulant property of the adjacency matrix allows the decision diagram to represent each column of the adjacency matrix under a well-defined permutation.

Another interesting application of these results is that well-known results in algebraic graph theory allow for the identification of bounds for the largest (and second largest) and smallest eigenvalue (in magnitude) to be determined based on properties of the adjacency matrix. This can lead to very useful results for techniques that require the use of the largest and smallest valued spectral coefficients (in magnitude) since such values can be bounded directly without searching through the entire spectrum of a function.

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