Spectral Transforms of Mixed-radix MVL Functions*

Mitchell A. Thornton
Department of Computer Science and Engineering
Southern Methodist University
Dallas, Texas
mitch@engr.smu.edu

Abstract

Mixed-radix "Multiple Valued Logic" (MVL) functions are assumed to be finite and discrete-valued and depend on a finite-valued variable support set $\{x_i, \ldots, x_j\}$ such that x_i is q_i-valued and x_j is q_j-valued with $q_i \neq q_j$. The spectra of such MVL functions is of interest to circuit designers and automated design tool researchers and developers. Spectral transforms are described that are applicable to such functions over the elementary additive (mod(p)) Abelian groups. Three formulation of such transforms are described here; a linear transformation matrix derived from a group character table, a Kronecker-based expansion allowing for a 'fast' transform algorithm, and a Cayley graph spectrum computation. It is shown that a particular spectral transformation of a discrete mixed-radix function over $\mathbf{Z_6}$ is equivalent to that over $\mathbf{Z_2} \times \mathbf{Z_3}$ within a permutation. Also, it is shown that a Cayley graph may be formed over $\mathbf{Z_6}$ with a generator corresponding to the discrete function of interest.

1 Introduction

It has been shown that the Walsh spectrum of a binary-valued function $f(x_1, x_2, \ldots, x_n)$ may be computed as the spectrum of a Cayley graph over the elementary additive Abelian group $\mathbf{Z_{2^n}}$ using a generator based on f [3]. These results were also generalized to a technique to compute the Chrestenson spectrum of finite discrete-valued functions in [13]. In general, these techniques can be proven to yield these spectra through the use of group character theory as described in [2] where the resulting spectral values are shown to be equivalent to inner products of the Cayley graph color vectors and the rows of the group character tables. If a proper generator is used in the formation of the Cayley graph, the corresponding color graph vector can be defined such that it is equivalent to a discrete function truth vector. Because the rows of the Walsh and Chrestenson transformation matrices are defined as the rows of the group character tables describing the elementary additive Abelian group [8], the spectrum of the Cayley color graph is equivalent to the spectrum of the discrete function.

The results in [3, 13] considered the spectra of p-valued functions with variable support sets consisting of q-valued variables. Here the subject of computing the spectra of p-valued functions with support sets of variables $\{x_1, x_2, \ldots, x_n\}$ are considered such that each variable x_i is q_i-valued and $q_i \neq q_j$ for various pairs of (i, j). Here, such functions are referred to as *mixed-radix* since the support variables are $x_i \in \mathbf{Z_i}$ for different values of i. It is noted that the case of the function f being p-valued is not as of much interest as the case of different-valued variables in the support set since for $p = 2$ a Cayley graph results and for $p > 2$ a Cayley color graph results which can simply be considered as a disjoint set of Cayley graphs.

In the remainder of this paper, mixed-radix transformations are considered and reviewed using transformation matrices formed as Kronecker products of elementary Walsh and Chrestenson matrices. It is also demonstrated that such transforms may be specified as a group character table over an appropriately ordered group and that the Cayley graph technique as described in [3, 13] may be generalized for the mixed-radix case.

2 Transformations of MVL Functions Over an Additive Abelian Group

The simplest case of a mixed-radix function where $q_i \geq 2$ is the binary-valued function $f(x_1, x_2)$ where $q_1 = 2$ and $q_2 = 3$. A straight forward extension of the techniques described in [3, 13] is to encode each minterm describing f as a unique element in $\mathbf{Z_6}$ and then to formulate the group character table for the additive Abelian group $mod(6)$ and utilize this table as a transformation matrix. This involves mapping each minterm to one of six roots of unity in the set $\{e^{j2\pi\cdot\frac{0}{6}}, e^{j2\pi\cdot\frac{1}{6}}, e^{j2\pi\cdot\frac{2}{6}}, e^{j2\pi\cdot\frac{3}{6}}, e^{j2\pi\cdot\frac{4}{6}}, e^{j2\pi\cdot\frac{5}{6}}\}$. A graphical depiction of these points is shown in Figure 1. An in-depth survey of the definition of discrete transforms using finite groups is available in [10]. Such transforms have been considered in the past [1, 8] and referred to as the *generalized transform* or *generalized Fourier transform*; however, the formulation and relationship of these discrete function spectra

*This project was supported by the NSF under grants CCR-0243358 and CCR-0243365.

to graph spectra offers a new viewpoint. It is noted that the formulation of the generalized Fourier transform typically includes the use of scaling constants referred to as "twiddling" factors. These are not included in the following without loss of generality.

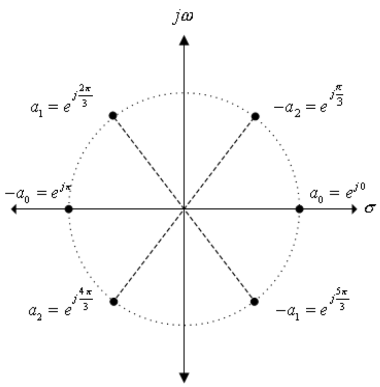

Figure 1. Diagram of Six Roots of Unity

A group character table is written with rows corresponding to irreducible representations and columns corresponding to conjugacy classes [6]. Traditionally the rows are labeled with χ_i which represent the irreducible representations as is done here. Each column (i.e. the conjugacy classes) is labeled with the notation as shown in Table 1. The purpose for this notation should become apparent later in this section of the paper.

Table 1. Group Character Table Over $\mathbf{Z_6}$

	x_0	x_1	x_2	x_3	x_4	x_5
χ_0	a_0	a_0	a_0	a_0	a_0	a_0
χ_1	a_0	$-a_2$	a_1	$-a_0$	a_2	$-a_1$
χ_2	a_0	a_1	a_2	a_0	a_1	a_2
χ_3	a_0	$-a_0$	a_0	$-a_0$	a_0	$-a_0$
χ_4	a_0	a_2	a_1	a_0	a_2	a_1
χ_5	a_0	$-a_1$	a_2	$-a_0$	a_1	$-a_2$

Using the results of [2], a transformation matrix may be formulated using the rows of the group character table as shown in Table 1 as rows in a transformation matrix. This can be accomplished using the so-called "R-encoding" where the transformation matrix contains complex-valued elements or with "S-encoding" where both the transformation matrix and the function truth vector are encoded into complex values [13]. In both cases, a linear transformation, vector-matrix product may be formulated and computed.

2.1 Kronecker Product Formulation

An alternative way of computing the transformation matrix for a function $f(x_1, x_2)$ where $x_1 \in \mathbf{Z_2}$ and $x_2 \in \mathbf{Z_3}$ is to utilize the Kronecker (or tensor) product [7] to combine the transformation matrices for the elementary additive Abelian group $mod(2)$ and the elementary additive Abelian group $mod(3)$ (e.g. note that these are also known as the Walsh and Chrestenson transformation matrices of functions of one variable). Equation 1 illustrates the computation of the transformation matrix T where $a_0 = e^{j2\pi \cdot \frac{0}{6}}$, $a_1 = e^{j2\pi \cdot \frac{2}{6}}$ and $a_2 = e^{j2\pi \cdot \frac{4}{6}}$.

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bigotimes \begin{bmatrix} a_0 & a_0 & a_0 \\ a_0 & a_1 & a_2 \\ a_0 & a_2 & a_1 \end{bmatrix} \quad (1)$$

Carrying out the calculation in Equation 1, the transformation matrix T becomes that as shown in Equation 2.

$$T = \begin{bmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & a_0 \\ a_0 & a_1 & a_2 & a_0 & a_1 & a_2 \\ a_0 & a_2 & a_1 & a_0 & a_2 & a_1 \\ a_0 & a_0 & a_0 & -a_0 & -a_0 & -a_0 \\ a_0 & a_1 & a_2 & -a_0 & -a_1 & -a_2 \\ a_0 & a_2 & a_1 & -a_0 & -a_2 & -a_1 \end{bmatrix} \quad (2)$$

It is easily shown that the transformation matrix T is unique as follows.

Lemma 1 *The inverse of matrix T exists and it is orthogonal (with a scale factor of $\frac{1}{2^n \cdot 3^n}$).*

Proof: In Equation 1, T is formed as the Kronecker product of the 2×2 Walsh transformation matrix W and the 3×3 Chrestenson transformation matrix C. Therefore the inverse of T is given as shown.

$$T^{-1} = (W \otimes C)^{-1} = W^{-1} \otimes C^{-1} \quad (3)$$

It is known that the inverses of W and C are given as follows.

$$W^{-1} = \frac{1}{2^n} W \quad (4)$$

$$C^{-1} = \frac{1}{3^n} C^* \quad (5)$$

Therefore T^{-1} is given as shown in Equation 6.

$$T^{-1} = \frac{1}{2^n \cdot 3^n} (W \otimes C^*) \quad (6)$$

□

Several other properties are noted with respect to the matrix T and are given in the following lemmas.

Lemma 2 *A linear transformation matrix formed using the rows and columns of the group character table given in Table 1 may is identical to matrix T under a set of row and column permutations.*

Proof: Let G represent the matrix formed using the rows and columns of the group character table given in Table 1. Equation 7 holds where U_1 and U_2 are elementary permutation matrices.

$$G = U_1 T U_2 \tag{7}$$

The matrices U_1 and U_2 are given in Equations 8 and 9.

$$U_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{9}$$

□

This technique is computationally advantageous to the other methods presented here since the decomposition of the transformation matrix into a Kronecker product of smaller dimensioned matrices allows for "fast-transform" techniques to be applied [5, 11]. As has been done traditionally, a butterfly diagram may be used to characterize the "fast-transform". A butterfly diagram is a signal flow graph where vertices represent summation operations and edges carry multiplicative weights. The butterfly diagram corresponding to Equation 1 is shown in Figure 2.

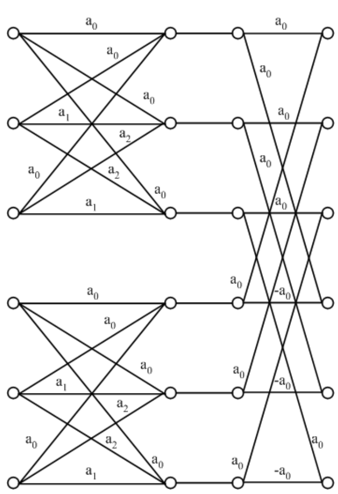

Figure 2. Butterfly Diagram of Fast Transform

Alternatively, Equation 1 may be rearranged using the permutation operations described in Lemma 2. When the permutation matrices are not included, the same spectral vector components result but in a different order. Equation 10 contains the relationship describing the Kronecker expansion of the transformation matrix in this form and Figure 3 is a an illustration of the corresponding butterfly diagram.

$$T = \begin{bmatrix} a_0 & a_0 & a_0 \\ a_0 & a_1 & a_2 \\ a_0 & a_2 & a_1 \end{bmatrix} \bigotimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{10}$$

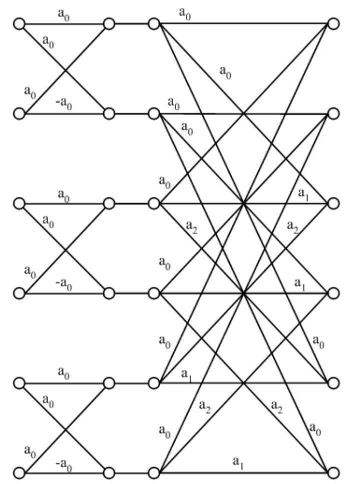

Figure 3. Alternative Butterfly Diagram of Fast Transform

Additionally such "fast-transform" methods may be implemented in a graphical manner using decision diagram data structures resulting in further savings [12, 14]. Although the minimum number of required operations is documented and has been studied in detail [4, 9], decision diagrams can offer an advantage by avoiding the cases of addition with a constant-0 and multiplication by a constant-1 since these operations are implicit due to the reduction rules of the data structures.

2.2 Cayley Graph Spectrum Formulation

To generalize the methods described in [3] for the binary-valued case and [13] for the discrete p-valued case where $p > 2$, a Cayley graph must be specified with an appropriate generator over the group $\mathbf{Z_{q_0 \cdot q_1 \cdot \ldots \cdot q_{n-1}}}$ where each variable x_i is q_i-valued.

The following definitions are used:

Definition 1 *A* Cayley graph *represents an algebraic group $G = \{g_i, *\}$ and is denoted as $Cay(V, E)$ where V is the vertex set and E is the edge set. Each $v_i \in V$ uniquely corresponds to the group element g_i. Each $e_i \in E$ corresponds to a colored edge with some associated color g_i. The set of edges E is a subset of ordered pairs of elements in V that are* generated *by some binary operation(s) over elements in G. This binary operator need not be the same as the group product operator $*$.*

In [3, 13] Cayley graphs were formed representing the elementary additive Abelian groups with generators that were the evaluation of some discrete function of the same group. The function argument was formulated as the digit-by-digit modulo-p difference of all possible minterms. For the binary case, the generator function yielding the adjacency matrix edge colors is $a_{ij} = f(m_i \oplus m_j)$ for all possible pairs of function domain values (m_i, m_j). Likewise for the non-binary p-valued case, adjacency matrix edge colors are generated as

$a_{ij} = f(m_i \ominus_p m_j)$. $\ominus_p$ denotes the digit-by-digit difference of two minterms modulo-p.

The generalization to the mixed-radix case is quite natural. If each minterm is composed of various polarities of n different q-valued variables, then argument of the generator function f is formed as a concatenation of $(x_{i1} \ominus_{q_1} x_{j1}), (x_{i2} \ominus_{q_2} x_{j2}), \ldots, (x_{in} \ominus_{q_n} x_{jn})$. This operation will be denoted by the symbol $\ominus$ with no subscript, but with the understanding that it is applied digit-by-digit and modulo-q_i. In this case the generator is given in Equation 11.

$$a_{ij} = f(m_i \ominus m_j) \qquad (11)$$

2.3 Example Spectrum Computation

Using Equation 11, the adjacency matrix for the Cayley graph over $\mathbf{Z_6}$ is given in Equation 12.

$$A = \begin{bmatrix} f(0) & f(2) & f(1) & f(3) & f(5) & f(4) \\ f(1) & f(0) & f(2) & f(4) & f(3) & f(5) \\ f(2) & f(1) & f(0) & f(5) & f(4) & f(3) \\ f(3) & f(5) & f(4) & f(0) & f(2) & f(1) \\ f(4) & f(3) & f(5) & f(1) & f(0) & f(2) \\ f(5) & f(4) & f(3) & f(2) & f(1) & f(0) \end{bmatrix} \qquad (12)$$

As an example consider the mixed-radix function described in Table 2.

Table 2. Truth-table of Example Function for Computation of Spectrum

x_1	x_2	X	*Mapping*	f
0	0	0	0	0
0	1	1	1	1
0	2	2	2	1
1	0	3	3	0
1	1	4	4	1
1	2	5	5	0

The corresponding adjacency matrix is given as shown by Equation 13 and an illustration of the Cayley graph is shown in Figure 4.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad (13)$$

The characteristic equation of A is $C(\lambda) = \lambda^6 - 6\lambda^4 - 10\lambda^3 + 6\lambda + 9$. Solving for the roots of $C(\lambda)$, the spectrum of f is found to be $\lambda_i = \{a_0, 3a_0, a_1, a_2, 2a_1 + a_2, a_1 + 2a_2\}$. It is easily verified that the same set of spectral coefficients result when the truth vector of f is multiplied with the transformation matrix given in Equation 2 or the one formed from Table 1.

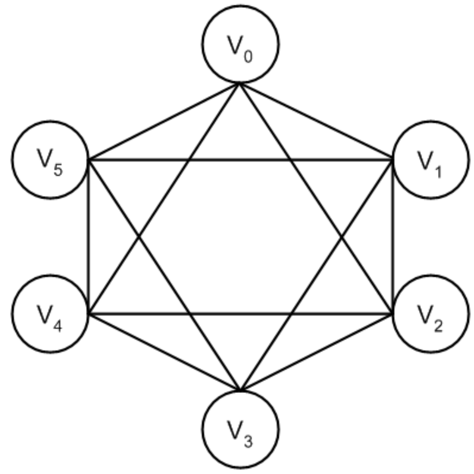

Figure 4. Cayley Graph of Example Function

3 Conclusion

It has been shown that a particular spectral transformation of a discrete mixed-radix function over $\mathbf{Z_6}$ is equivalent to that over $\mathbf{Z_2} \times \mathbf{Z_3}$ within a permutation. Also, it is shown that a Cayley graph may be formed over $\mathbf{Z_6}$ with a generator corresponding to the discrete function of interest. The spectrum of the Cayley graph is equivalent to the spectrum of the discrete mixed-radix function.

References

[1] N. Ahmed and K. R. Rao. *Orthogonal Transforms for Digital Signal Processing*. Springer-Verlag, New York, New York, 1975.

[2] L. Babai. Spectra of Cayley graphs. *Journal of Combinatorial Theory*, Series B, 27:180–189, 1979.

[3] A. Bernasconi and B. Codenotti. Spectral analysis of Boolean functions as a graph eigenvalue problem. *IEEE Trans. on Comp.*, 48:345–351, 1999.

[4] M. Clausen. Fast generalized Fourier transforms. *Theoretical Computer Science*, 67:55–63, 1989.

[5] J. W. Cooley and J. W. Tukey. An algorithm for the machine calculation of complex fourier series. *Math. Computation*, 19:297–301, 1965.

[6] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Prentice Hall, Upper Saddle River, New Jersey, 1999.

[7] A. Graham. *Kronecker Products and Matrix Calculus: with Applications*. Ellis Horwood Limited and John Wiley & Sons, New York, 1981.

[8] M. Karpovsky. *Finite Orthogonal Series in the Design of Digital Devices*. Wiley and JUP, 1976.

[9] D.K. Maslen. The efficient computation of Fourier transforms on the symmetric group. *Math. Comp.*, 67(223):1121–1147, 1998.

[10] D.K. Maslen and D.N. Rockmore. Generalized FFTs - A survey of some recent results. In *DIMACS Workshop in Groups and Computation*, pages 183–238, 1995.

[11] J. L. Shanks. Computation of the fast Walsh-Fourier transform. *IEEE Trans. on Comp.*, 18:457–459, 1969.

[12] R. S. Stanković, T. Sasao, and C. Moraga. Spectral transforms decision diagrams. In T. Sasao and M. Fujita, editors, *Representation of Discrete Functions*, pages 55–92. Kluwer Academic Publishers, 1996.

[13] M. A. Thornton, D. M. Miller, and W. Townsend. Chrestenson spectrum computation using Cayley color graphs. In *Int'l Symp. on Multi-Valued Logic*, pages 123–128, 2002.

[14] M.A. Thornton, R. Drechsler, and D.M. Miller. *Spectral Techniques in VLSI CAD*. Kluwer Academic Publishers, 2001.