Composition Methods for Four-Port Couplers in Photonic Integrated Circuitry

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Abstract

Planar photonic integrated circuits based on four-port couplers offer enhanced sophistication and functionality. Each four-port coupler is characterized by sixteen signal coupling coefficients governed by ten energy constraints. The ability to generate the constrained sixteen coupling coefficients is needed in the analysis of the four-port coupler. However, the energy constraint equations are nonlinear and cumbersome to solve directly. We introduce two techniques to reduce these signal coupling coefficients to a set of six free parameters. Hence we can characterize all possible couplers in terms of their sixteen constrained coupling coefficients, or either of two sets of six free parameters. This reduction in parameters has significant ramifications for the design, specification and empirical characterization of these useful building blocks.

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1. Introduction

A fundamental building block of photonic integrated circuitry is the coupler used to split and combine optical signals. Traditional linear, one dimensional lasers and lattice filters are enabled by two-port couplers characterized by reflection and transmission coefficients. Because of their wide utility, the design and analysis of structures comprising large numbers of these partially reflecting and partially transmitting mirrors is well studied [1-10]. In planar photonic architectures, the four-port coupler enables lasers, filters and hybrid circuits with increasing sophistication and functionality [11-14].

In modern photonic integrated circuits the four-port coupler is precise to the nanometer scale, and couples signals across four input/output waveguides. These waveguides may include gain, for example from a semiconductor optical amplifier, or they may be passive [9-11]. Figure 1 shows an example of a four-port coupler fabricated by focused ion beam milling in an active filter constructed using semiconductor optical amplifiers. In most configurations these four waveguides are designed to the single mode and typically support one polarization. Interestingly, in practice these devices are often flat over a reasonable frequency range, for example across the C-band or L-band of erbium doped fiber amplifier telecommunication systems. Under these conditions the four-port coupler may be described by a 16 element scattering matrix, and it is this case that is considered below.

Figure 2 shows the signal flow diagram for the four-port coupler under consideration here. The coupler may support up to four input signals, and will yield four output signals. In this paper, the four ports are referred to as N, S, E and W. At any given port, a portion of the entering signal may be transmitted, reflected, routed to the right, or routed to the left. Thus, from a functional point of view there are sixteen coefficients relevant to the design and the description of a four-port coupler. Energy conservation, however, provides ten constraints associated with these sixteen coefficients [11]. In fact, a four-port coupler can be represented by a 4x4 matrix that is orthogonal in the sense that the product of the matrix and its transpose is the identity matrix. Relevant to the discussion below, we note that a real orthogonal matrix is a special case of a complex unitary matrix, for which the product of the matrix and its conjugate transpose is the identity matrix. In the microwave literature, this 4x4 matrix is the well known scattering, or S-matrix [15, 16]. The study of four-port couplers and the networks they enable have a long and useful history [17-26].

The researchers are developing nanoscale photonic four port couplers for use in active and passive tunable filters comprising a multiplicity of these elements [11]. In the filter theory, it has proven important to understand the set of available couplers to include in the overall design space. The unitary nature of the S-matrix greatly restricts the realized space for these elements and the filters that rely upon them. This paper provides novel and useful approaches to implanting these constraints on the four-port couplers.

In principle, the ten constraints suggest six independent degrees of freedom. Thus, a four-port, lossless coupler may be characterized by six well chosen measurements. However, the ten constraints result from the dot products between columns of the S-matrix and are nonlinear and hence cumbersome. A major problem in determining the parameters of a four-port coupler is that choosing sixteen parameters that satisfy the ten constraints by inspection is difficult, or, equivalently, choosing the entries of a 4x4 orthogonal matrix by trial and error is daunting. This paper presents two approaches to solve this problem. Both approaches have implications in measurement and characterization of four-port couplers [27, 28]. A procedure that works in special cases was introduced in [29].

The first method, described in Section 2, uses the Cayley map [30, 31] and represents an orthogonal matrix in terms of a 4x4 skew symmetric matrix (in which an element is the negative of its value under transpose). Given a skew symmetric matrix, F, or equivalently, an arbitrary set of six real numbers, we can generate an orthogonal matrix using the Cayley map. Conversely, the inverse Cayley map can be applied to generate a skew symmetric matrix from a given orthogonal matrix. The symbolic program Mathematica is applied to make the representation of an orthogonal matrix in terms of the six parameters more explicit than the general formulas.

The second method, found in Section 3 provides an algorithmic decomposition of an orthogonal matrix, or, more generally, a unitary matrix, of any size (with particular interest in 4x4 matrices for our purposes) [32]. In the case of an orthogonal matrix the decomposition is the product of simple orthogonal matrices depending on a reduced number of parameters and plus or minus signs. For example, in the 4x4 orthogonal case the orthogonal matrix is the product of ten 4x4 matrices, six of which include one parameter each, and the other four representing a choice of signs. Hence, given any six real parameters and a choice of signs, we can generate a unique orthogonal matrix, and hence a unique four-port coupler. Conversely, given an orthogonal matrix or four-port coupler, we can determine the six parameters and the associated signs. Decomposition into products of simple orthogonal matrices allows representation of orthogonal matrices as equivalent transmission lines. Similar decompositions hold for unitary matrices. This transmission line description of an orthogonal (or unitary matrix) leads to a combinatorial approach using the Dyck triangle and Catalan numbers [33] shown in the Appendix.

2. The Cayley Mapping

Given the physical description of the device, energy conservation arguments can be used to provide a mathematical description of the four-port coupler shown in Figure 2. The device has four input/output ports designated by North, South, East, and West. Each port has a reflection coefficient, ρ, a transmission coefficient, τ, a right handed coupling coefficient, α, and a left handed coupling coefficient, β. Given these coefficients, a device can be uniquely represented by a scattering matrix, *S*, of these sixteen parameters. For a realizable, stable system, this scattering matrix must conserve energy such that the input energy equals the output energy. To enforce these energy constraints, we declare that the scattering matrix, *S* be a unitary matrix. Since all of the elements of the scattering matrix are real, it is sufficient to say that an orthogonal scattering matrix will satisfy the energy constraints.

We first show that for each orthogonal matrix there is a unique real 4x4 skew symmetric matrix, consisting of six independent parameters, generating the given orthogonal matrix and vice versa. Hence we can categorize all four-port couplers that can be built.

There are at least two ways to generate orthogonal matrices out of skew-symmetric matrices: 1) the Cayley Transform [30, 31] and 2) the exponential mapping [30]. We prefer the former since it is easier to invert.

In any dimension, if F is n x n real and skew-symmetric (i.e., $F^T = -F$), then $S = (I+F)(I-F)^{-1} = (I-F)^{-1}(I+F)$ is real and special orthogonal (i.e. det(S)=1). Conversely, every *S* that does not have -1 as one of its eigenvalues can be so realized. From this point onwards the discussion in this section pertains to 4x4 orthogonal matrices.

The above definition of the Cayley mapping is slightly different from the customary definition. If *F* were to be replaced by -*F* then one would recover the usual definition. However, since -*F* is skew-symmetric if and only if *F* is, the above definition is correct for the purposes of parameterizing orthogonal matrices by six parameters.

To also cover the case of i) special orthogonal matrices that have -1 as an eigenvalue; and ii) orthogonal matrices with determinant -1, we can proceed as follows. For case i) we introduce an extra parameter k and study instead the modified version of the Cayley mapping, viz., $(kI+F)(kI-F)^{-1}$. While this parameterization ostensibly yields seven parameters as opposed to the six, it is in principle a six dimensional parameterization. Indeed, if (k,F) represent S, then so do (ck,cF) for every $c \neq 0$. The technique for computing $(I+F)(I-F)^{-1}$ extends verbatim to computing $(kI+F)(kI-F)^{-1}$. However, since the -1 eigenvalue case is non-generic (the section on the inverse Cayley mapping will also yield a characterization of such matrices), we deal with just mapping $S \rightarrow (I+F)(I-F)^{-1}$.

For case ii) we note that every orthogonal matrix with determinant -1 may be written as DF with D=diag(-1,1,1,1) and F special orthogonal. So it suffices to consider only special orthogonal F.

We now present an explicit formula for computing $(I+F)(I-F)^{-1}$, which avoids the inversion implicit in the Cayley mapping. Prior to doing that two vectors in $\Re^3$ will be associated with every 4x4 skew-symmetric F. These vectors are defined in the following fashion.

Let

$$F = \begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & f_4 & f_5 \\ -f_2 & -f_4 & 0 & f_6 \\ -f_3 & -f_5 & -f_6 & 0 \end{bmatrix} \qquad (1)$$

Then, define:

$$s \in \Re^3 = \frac{1}{2}\left[-(f_1+f_6),(f_5-f_2),-(f_3+f_4)\right]$$

$$t \in \Re^3 = \frac{1}{2}\left[(f_1 - f_6),(f_5 + f_2),(f_3 - f_4)\right]$$

The following scalar quantities will also be used

$$\lambda^2 = \left(\|s\|^2 + \|t\|^2\right) \qquad \mu = \left(\|s\|^2 - \|t\|^2\right)$$

where || · || denotes the usual norm on $\mathbb{R}^3$. For non-zero *S*, three cases arise:

Case I: Either *s* or *t* is zero. In this case the minimal polynomial of *F* is $F^2 + \lambda^2 I_4 = 0$. Therefore, *S* is expressible as *AI*+*BF*. After comparing like powers of *F*, the formula for *S* is

$$S = \frac{1}{1+\lambda^2}\left[\left(1-\lambda^2\right)I + 2F\right] \tag{2}$$

Case II: If $\|s\| = \|t\|$, then the minimal polynomial of *F* is cubic with $F^3 = -4\|s\|^2 F$. Once more, *S* is

$$S = I + \frac{2}{1+4\|s\|^2}\left(F + F^2\right) \tag{3}$$

Case III: The generic case, neither case I nor case II.

$$S = \frac{1}{1+2\lambda^2+\mu^2}\left[\left(1+2\lambda^2-\mu^2\right)I + \left(2+4\lambda^2\right)F + 2F^2 + 2F^3\right] \tag{4}$$

From the Cayley mapping the sixteen parameter, orthogonal scattering matrix *S* can be represented in terms of six unique parameters given in the upper triangle (or inverted lower triangle) of the skew-symmetric matrix *F*.

Using the previously defined Cayley mapping formulas the scattering matrix *S* can be redefined in terms of the six unique parameters $f_{1\text{-}6}$. Considering the three cases above and employing Mathematica:

Case I:

$$S = \frac{2}{1+\lambda^2}\begin{bmatrix} 1-\lambda^2 & f_1 & f_2 & f_3 \\ -f_1 & 1-\lambda^2 & f_4 & f_5 \\ -f_2 & -f_4 & 1-\lambda^2 & f_6 \\ -f_3 & -f_5 & -f_6 & 1-\lambda^2 \end{bmatrix} \tag{5}$$

Case II:

$$S = \Gamma\begin{bmatrix} A & B & C & D \end{bmatrix} \tag{6}$$

where

$$A = \begin{bmatrix} (\frac{1}{\Gamma} - f_1^{\ 2} - f_2^{\ 2} - f_3^{\ 2}) \\ (f_1 - f_2 f_4 - f_3 f_5) \\ (f_2 + f_1 f_4 - f_3 f_6) \\ (f_3 + f_2 f_6 + f_1 f_5) \end{bmatrix}$$

$$B = \begin{bmatrix} (-f_1 - f_2 f_4 - f_3 f_5) \\ (\frac{1}{\Gamma} - f_1^{\ 2} - f_4^{\ 2} - f_5^{\ 2}) \\ (f_4 - f_1 f_2 - f_5 f_6) \\ (f_5 - f_1 f_3 + f_4 f_6) \end{bmatrix}$$

$$C = \begin{bmatrix} (-f_2 + f_1 f_4 - f_3 f_6) \\ (-f_4 - f_1 f_2 - f_5 f_6) \\ (\frac{1}{\Gamma} - f_2^{\ 2} - f_4^{\ 2} - f_6^{\ 2}) \\ (f_6 - f_2 f_3 - f_4 f_5) \end{bmatrix}$$

$$D = \begin{bmatrix} (-f_3 + f_2 f_6 + f_1 f_5) \\ (-f_5 - f_1 f_3 + f_4 f_6) \\ (-f_6 - f_2 f_3 - f_4 f_5) \\ (\frac{1}{\Gamma} - f_3^{\ 2} - f_5^{\ 2} - f_6^{\ 2}) \end{bmatrix}$$

and

$$\Gamma = \frac{2}{1+4\|s\|^2}$$

Case III:

$$S = \mathrm{E}\begin{bmatrix} K & L & M & N \end{bmatrix} \tag{7}$$

where

$$K = \begin{bmatrix} \frac{-1}{\mathrm{E}} + 1 + f_4^{\,2} + f_5^{\,2} + f_6^{\,2} \\ -(f_1 f_6^{\,2} + f_1 + f_2 f_4 + f_3 f_5 + f_3 f_4 f_6 - f_2 f_5 f_6) \\ -(f_2 f_5^{\,2} + f_2 - f_1 f_4 + f_3 f_6 - f_3 f_4 f_5 - f_1 f_5 f_6) \\ -(f_3 f_4^{\,2} + f_3 - f_2 f_6 - f_1 f_5 - f_2 f_4 f_5 + f_1 f_4 f_6) \end{bmatrix}$$

$$L = \begin{bmatrix} (f_1 f_6^{\,2} + f_1 - f_2 f_4 - f_3 f_5 + f_3 f_4 f_6 - f_2 f_5 f_6) \\ \frac{-1}{\mathrm{E}} + 1 + f_2^{\,2} + f_3^{\,2} + f_6^{\,2} \\ -(f_4 f_3^{\,2} + f_4 + f_1 f_2 + f_5 f_6 + f_1 f_3 f_6 - f_2 f_3 f_5) \\ -(f_5 f_2^{\,2} + f_5 + f_1 f_3 - f_4 f_6 - f_2 f_3 f_4 - f_1 f_2 f_6) \end{bmatrix}$$

$$M = \begin{bmatrix} (f_2 f_5^{\,2} + f_2 + f_1 f_4 - f_3 f_6 - f_3 f_4 f_5 - f_1 f_5 f_6) \\ -(f_4 f_3^{\,2} - f_4 + f_1 f_2 + f_5 f_6 - f_1 f_3 f_6 + f_2 f_3 f_5) \\ \frac{-1}{\mathrm{E}} + 1 + f_1^{\,2} + f_3^{\,2} + f_5^{\,2} \\ -(f_6 f_1^{\,2} + f_6 + f_2 f_3 + f_4 f_5 + f_1 f_3 f_4 - f_1 f_2 f_5) \end{bmatrix}$$

$$N = \begin{bmatrix} (f_3 f_4^{\,2} + f_3 + f_2 f_6 + f_1 f_5 + f_1 f_4 f_6 - f_2 f_4 f_5) \\ -(-f_5 f_2^{\,2} - f_5 + f_1 f_3 - f_4 f_6 + f_2 f_3 f_4 + f_1 f_2 f_6) \\ -(-f_6 f_1^{\,2} - f_6 + f_2 f_3 + f_4 f_5 - f_1 f_3 f_4 + f_1 f_2 f_5) \\ \frac{-1}{\mathrm{E}} + 1 + f_1^{\,2} + f_2^{\,2} + f_4^{\,2} \end{bmatrix}$$

and

$$\mathrm{E} = \frac{2}{1 + 2\lambda^2 + \mu^2}$$

With this translation in place, an arbitrary scattering matrix can be generated. Given any six parameters, it is guaranteed that the matrix created using the definitions above will satisfy all ten constraining equations, i.e., the matrix created is guaranteed to be orthogonal.

We now provide examples where we take values from $f_1 - f_6$ and show the resulting orthogonal matrices.

$$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{2}{3}$$

$$G = \begin{bmatrix} 0.2077 & 0.0383 & 0.1917 & 0.9585 \\ -0.9585 & 0.2077 & 0.0383 & 0.1917 \\ -0.1917 & -0.9585 & 0.2077 & 0.0383 \\ -0.0383 & -0.1917 & -0.9585 & 0.2077 \end{bmatrix}$$

$$f_1 = f_2 = f_3 = 0$$
$$f_4 = f_5 = f_6 = \frac{1}{2}$$
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4286 & 0.2857 & 0.8571 \\ 0 & -0.8571 & 0.4286 & 0.2857 \\ 0 & -0.2857 & -0.8571 & 0.4286 \end{bmatrix}$$

Inverse Cayley Mapping

The forward Cayley mapping was defined via $\phi(F) = (I+F)(I-F)^{-1}$. The image of ϕ is in $SO(4,R)$ (4x4 special orthogonal matrices) and misses matrices with -1 as an eigenvalue. For $G \in SO(4,R)$, without -1 as an eigenvalue the inverse Cayley mapping is $\psi(S) = (I+S)^{-1}(S-I)$. We explain how to find Ψ without doing the requisite inversion.

First, any such S can be represented by a pair of unit quaternions [34] $p = p_0 + p_1 i + p_2 j + p_3 k = p_o 1 + \tilde{p}$, $q = q_0 + q_1 i + q_2 j + q_3 k = q_0 1 + \tilde{q}$. Let us defer the formulae for p and q in terms of the entries of S for the moment.

Next, the characteristic polynomial of S is $\lambda^4 - Tr(S)\lambda^3 + \kappa\lambda^2 Tr(S)\lambda + 1$. The quantities Tr($S$) and κ are found as follows:

$$Tr(S) = 4p_0q_0 \quad (8)$$

$$\kappa = 6p_0{}^2q_0{}^2 + 2p_0{}^2\tilde{q}\cdot\tilde{q} - 2(\tilde{q}\cdot\tilde{q})(\tilde{p}\cdot\tilde{p}) \quad (9)$$

The eigenvalues of S are of the form $e^{i\theta}, e^{-i\theta}$, $e^{i\alpha}, e^{-i\alpha}$. These can be found by inspection. Without loss of generality, suppose cosθ ≥ cosα. Let x = cosθ, y = cosα. Then $x + y = 2p_0q_o$ and $x - y = \frac{\kappa}{2} - 1$.

This yields x and y and hence the eigenvalues of S. Since S is diagonalizable, the algebraic multiplicity of the eigenvalues determines the minimal polynomial of S. We have:

Case I: $S = I$;

Case II: $e^{i\theta} = e^{i\alpha}$ or $e^{-i\alpha}$ iff x = y iff $p_o{}^2q_o{}^2 = \kappa - 2$. In this case, S has a quadratic polynomial, which is $S^2 - 2p_oq_o + I = 0$.

Case III: $p_0 = q_o = 0$. In this case $S^2 = I$ (Note: this means S is symmetric, in addition to being orthogonal). However, this implies S has two eigenvalues equal to -1.

Case IV: $e^{i\theta} = 1$ and $e^{i\alpha} \neq 1, -1$ iff x = 1, -1 < y < 1 iff $2 = 2p_oq_o + (4p_o{}^2q_o{}^2 - \kappa + 2)^{1/2}$. In this case $S^3 - (4p_oq_o - 1)S^2 + (4p_oq_o - 1)S - I = 0$.

In each of these cases (omitting those where S has -1 as an eigenvalue) the explicit formula for the Inverse Cayley mapping is given by:

Case I: F=0

Case II: F=AI+BG, with $A = B - 1,\ B = \frac{1}{1 + p_oq_o}$

Case III: $F = AI + BG + CG^2$, with $A = -C - 1,\ B = 1,\ C = \frac{-1}{4p_oq_o}$

Case IV: The General Case (four distinct eigenvalues): $F = AI + BS + CS^2 + DS^3$ with

$$A = D - 1,\ B = -A + 1 - 4p_o q_o D,\ C = -B + D\kappa,\ D = \frac{2}{\kappa + 2 + 8p_o q_o}$$

Algorithmic Procedure to find p and q from S

Let S=[$x|y|u|v$], i.e., x,y,u, and v are the columns of S. Identify each of these columns with the corresponding unit quaternions, i.e., write $x = x_o + x_1 i + x_2 j + x_3 k$.

Compute the following three quaternions $V_1 = \tilde{x}y,\ V_2 = \tilde{x}u,\ V_3 = \tilde{x}v$ (Note: do not confuse these V_i's with the components of v). These three quaternions will have no real part, and can be identified with three vectors in $\mathbb{R}^3$.

The real 3x3 matrix $H = [V_1 \mid V_2 \mid V_3]$ takes the following form

$$H = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2ab+2cd & a^2-b^2-c^2+d^2 \end{bmatrix} \qquad (10)$$

where q=a+bi+cj+dk. This yields a system of equations

$$a^2 + b^2 + c^2 + d^2 = 1 \qquad (11)$$

$$a^2 + b^2 - c^2 - d^2 = h_{11} \qquad (12)$$

$$a^2 - b^2 + c^2 - d^2 = h_{22} \qquad (13)$$

$$a^2 - b^2 - c^2 + d^2 = h_{33} \qquad (14)$$

From these equations we get $a^2 + d^2;\ b^2 + c^2;\ a^2 - d^2;\ b^2 - c^2$. Hence, we get a, b, c, d up to the choice of sign, or parity.

Suppose $a \neq 0$. Pick one choice for the sign of a. The remaining entries of H yield ac, ad, ab (specifically h_{13}, h_{31} give ac, bd, h_{12}, h_{21} yield ad, bc; h_{23}, h_{32} yield ab, cd). With the choice of a, one can find the correct sign of b, c, d. If $a = 0$, suppose $b \neq 0$. Make a choice for the sign of b. With this choice, and the fact that bc, bd are known one can determine c, d correctly. If $a = b = 0$, suppose $c \neq 0$. Making a choice for the sign of c and knowing cd one finds d correctly. If $a = b = c = 0$, then $d \neq 0$. Pick one choice for the sign of d. This yields q up to sign (i.e., reversing the initial choice of sign would yield $-q$, and both are acceptable solutions). Once q is known $p = xq$.

We now provide an example for the inverse Cayley Mapping. This particular example is for a symmetric, lossless coupler that evenly distributes energy input at one port to each output port. The researchers have several detailed finite-difference time domain designs for this coupler and are developing the fabrication process in order to make a filter comprised of a field of these couplers interconnected by active or phase tunable waveguides. The orthogonal matrix

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

is generated by the pair of unit quaternions

$$p = 1 + 0i + 0j + 0k$$
$$q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$$

In this case S has a quadratic minimum polynomial (Case II). Hence F must take the form

$F = AI + BG$

where

$$B=\frac{1}{1+p_o q_o}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$$

$$A=B-1=-\frac{1}{3}$$

Thus $F=-\frac{1}{3}I+\frac{2}{3}S$ and

$$F=\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

3. Decomposition as a Product of Simple Unitary Matrices and as Transmission Lines

We now show that for each unitary (with orthogonal as a special case) matrix there is a unique 4 x 4 matrix decomposition, consisting of six independent parameters, generating the given orthogonal matrix and vice versa. All four-port couplers can be represented as a product of simple unitary matrices and as their transmission line equivalents. This decomposition applies to unitary matrices. The results are later specialized to real orthogonal matrices. We will begin by demonstrating the algorithm on a simple 2x2 unitary matrix, *P*. Next, we will apply the decomposition algorithm to the scattering matrix, *S*.

Let *U(n)* be the group of *n* x *n* unitary matrices (with complex entries). For illustration this analysis begins with *n* = 2.

$$\text{Let } P=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\in U(2)$$

Since $P^*P=I$, $|p_{11}|\le 1$, and setting $\gamma_{11}=p_{11}$, $p_{21}=\gamma_{21}d_{11}$ with $|\gamma_{21}|=1$.

For $|\gamma_{ij}| \leq 1$, we set $d_{ij} = \left(1 - |\gamma_{ij}|^2\right)^{\frac{1}{2}}$. Note that for $|p_{11}| = 1$, $p_{21} = 0$ and γ_{21} is not determined by *P*; we choose $\gamma_{21} = 1$ in this case. For $|\gamma_{ij}| \leq 1$, we set $J(\gamma_{ij}) = \begin{bmatrix} \gamma_{ij} & d_{ij} \\ d_{ij} & -\bar{\gamma}_{ij} \end{bmatrix}$. Multiply *P* on the left by $J(\gamma_{11})^* \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma}_{21} \end{bmatrix}$ and get

$$J(\gamma_{11})^* \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma}_{21} \end{bmatrix} P = \begin{bmatrix} \bar{\gamma}_{11} & d_{11} \\ d_{11} & -\gamma_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma}_{21} \end{bmatrix} \begin{bmatrix} \gamma_{11} & s_{12} \\ \gamma_{21} d_{11} & s_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\gamma}_{11} & d_{11} \\ d_{11} & -\gamma_{11} \end{bmatrix} \begin{bmatrix} \gamma_{11} & s_{12} \\ d_{11} & \bar{\gamma}_{21} s_{22} \end{bmatrix} = \begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix} \tag{14}$$

Since $\begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix} \in U(2)$, we must have *A* = 0 and |*B*| = 1, so that (with $\gamma_{12} = B$)

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_{21} \end{bmatrix} J(\gamma_{11}) \begin{bmatrix} 1 & 0 \\ 0 & \gamma_{12} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & d_{11}\gamma_{12} \\ \gamma_{21} d_{11} & -\gamma_{21}\bar{\gamma}_{11}\gamma_{12} \end{bmatrix}$$

The transmission line interpretation of the matrix multiplication is shown in Figure 3.

Since *P* is a 2x2 matrix, the two input/two output map is seen through matrix multiplication. However, if we have only the transmission line representation, then the first output as a function of the first input is the (1,1) entry of $J(\gamma_{11})$ times that input. Moreover, the first output as a function of the second input is the modulus one constant γ_{12} times the (1,2) entry of $J(\gamma_{11})$ times the input. Similarly, the second input as a function of the first input is the (2,1) entry of $J(\gamma_{11})$ times the modulus one constant γ_{21} times the input, and the second output as a function of the second input is the modulus one constant γ_{12} times the (2,2) entry of $J(\gamma_{11})$ times the modulus one constant γ_{21} times the input.

Now consider the case of the scattering matrix, *n* = 4.

Let $S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} \in U(4)$

As before, for $|\gamma_{ij}| \le 1$ we set

$$d_{ij} = \left(1 - |\gamma_{ij}|^2\right)^{\frac{1}{2}} \tag{15}$$

Case I: $|s_{11}| < 1$

Define

$$\gamma_{11} = s_{11} \tag{16}$$

and

$$\gamma_{21} = \frac{s_{21}}{d_{11}} \tag{17}$$

Case II: $|\gamma_{21}| < 1$

Define

$$\gamma_{31} = \frac{s_{31}}{d_{21} d_{11}} \tag{18}$$

Case III: $|\gamma_{31}| < 1$

Define

$$\gamma_{41} = \frac{s_{41}}{d_{31} d_{21} d_{11}} \tag{19}$$

Since $|s_{11}|^2 + |s_{21}|^2 + |s_{31}|^2 + |s_{41}|^2 = 1$, we must have $|\gamma_{41}| = 1$ (in the real case $\gamma_{41} = \pm 1$).

Multiplying S on the left by $S_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{\gamma}_{41} \end{bmatrix}$ we deduce

$$S' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{\gamma}_{41} \end{bmatrix} S = \begin{bmatrix} \gamma_{11} & s_{12}{}^{41} & s_{13}{}^{41} & s_{14}{}^{41} \\ \gamma_{21} d_{11} & s_{22}{}^{41} & s_{23}{}^{41} & s_{24}{}^{41} \\ \gamma_{31} d_{21} d_{11} & s_{32}{}^{41} & s_{33}{}^{41} & s_{34}{}^{41} \\ d_{31} d_{21} d_{11} & s_{42}{}^{41} & s_{43}{}^{41} & s_{44}{}^{41} \end{bmatrix}$$

Multiplying S on the left by $S_{31}{}^{*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & d_{31} \\ 0 & 0 & d_{31} & -\bar{\gamma}_{31} \end{bmatrix}^{*}$ we deduce

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\gamma}_{31} & d_{31} \\ 0 & 0 & d_{31} & -\gamma_{31} \end{bmatrix} \begin{bmatrix} \gamma_{11} & s_{12}{}^{31} & s_{13}{}^{31} & s_{14}{}^{31} \\ \gamma_{21} d_{11} & s_{22}{}^{31} & s_{23}{}^{31} & s_{24}{}^{31} \\ \gamma_{31} d_{21} d_{11} & s_{32}{}^{31} & s_{33}{}^{31} & s_{34}{}^{31} \\ d_{31} d_{21} d_{11} & s_{42}{}^{31} & s_{43}{}^{31} & s_{44}{}^{31} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_{11} & s_{12}{}^{31} & s_{13}{}^{31} & s_{14}{}^{31} \\ \gamma_{21} d_{11} & s_{22}{}^{31} & s_{23}{}^{31} & s_{24}{}^{31} \\ \left(|\gamma_{31}|^2 + d_{31}{}^2\right) d_{21} d_{11} & s_{32}{}^{31} & s_{33}{}^{31} & s_{34}{}^{31} \\ (d_{31}\gamma_{31} - \gamma_{31} d_{31}) d_{21} d_{11} & s_{42}{}^{31} & s_{43}{}^{31} & s_{44}{}^{31} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_{11} & s_{12}{}^{31} & s_{13}{}^{31} & s_{14}{}^{31} \\ \gamma_{21} d_{11} & s_{22}{}^{31} & s_{23}{}^{31} & s_{24}{}^{31} \\ d_{21} d_{11} & s_{32}{}^{31} & s_{33}{}^{31} & s_{34}{}^{31} \\ 0 & s_{42}{}^{31} & s_{43}{}^{31} & s_{44}{}^{31} \end{bmatrix}$$

By doing two more similar steps we deduce

$$S_{11}{}^{*} S_{21}{}^{*} S_{31}{}^{*} S_{41}{}^{*} S = S_1 = \begin{bmatrix} 1 & s_{12}{}^{1} & s_{13}{}^{1} & s_{14}{}^{1} \\ 0 & s_{22}{}^{1} & s_{23}{}^{1} & s_{24}{}^{1} \\ 0 & s_{32}{}^{1} & s_{33}{}^{1} & s_{34}{}^{1} \\ 0 & s_{42}{}^{1} & s_{43}{}^{1} & s_{44}{}^{1} \end{bmatrix}$$

Since S_1 is unitary, we must have $s_{12}{}^{1} = s_{13}{}^{1} = s_{14}{}^{1} = 0$, so

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s_{22}{}^1 & s_{23}{}^1 & s_{24}{}^1 \\ 0 & s_{32}{}^1 & s_{33}{}^1 & s_{34}{}^1 \\ 0 & s_{42}{}^1 & s_{43}{}^1 & s_{44}{}^1 \end{bmatrix} \tag{20}$$

Case IV: $|s_{22}{}^1| < 1$

Define

$$\gamma_{22} = s_{22}{}^1 \tag{21}$$

and

$$\gamma_{32} = \frac{s_{32}{}^1}{d_{22}} \tag{22}$$

Case V: $|\gamma_{32}| < 1$

Define

$$\gamma_{42} = \frac{s_{42}{}^1}{d_{32}d_{22}} \tag{23}$$

Since $\left|s_{22}{}^1\right|^2 + \left|s_{32}{}^1\right|^2 + \left|s_{42}{}^1\right|^2 = 1$, we must have $|\gamma_{42}| = 1$ (in the real case $\gamma_{42} = \pm 1$). Proceeding as before (note the superscript 2 does not denote the square operation),

$$S_{22}{}^* S_{32}{}^* S_{42}{}^* S_1 = S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s_{23}{}^2 & s_{24}{}^2 \\ 0 & 0 & s_{33}{}^2 & s_{34}{}^2 \\ 0 & 0 & s_{43}{}^2 & s_{44}{}^2 \end{bmatrix}$$

and since S_2 is unitary, $s_{23}{}^2 = s_{24}{}^2 = 0$, hence

$$S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s_{33}{}^2 & s_{34}{}^2 \\ 0 & 0 & s_{43}{}^2 & s_{44}{}^2 \end{bmatrix} \tag{24}$$

Case VI: $|s_{33}{}^2| < 1$

Define

$$\gamma_{33} = s_{33}{}^{2} \tag{25}$$

and
$$\gamma_{43} = \frac{s_{43}{}^{2}}{d_{33}} \tag{26}$$

with $|\gamma_{43}| = 1$.

As before,

$$S_{33}{}^{*}S_{43}{}^{*}S_{2} = S_{3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_{44}{}^{3} \end{bmatrix} \text{ with } \left|s_{44}{}^{3}\right| = 1 \text{ and } \gamma_{44} = s_{44}{}^{3}$$

We can now write

$$S = S_{41}S_{31}S_{21}S_{11}S_{42}S_{32}S_{22}S_{43}S_{33}S_{44} \tag{27}$$

or

$$GS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{41} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{31} & d_{31} \\ 0 & 0 & d_{31} & -\bar{\gamma}_{31} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{21} & d_{21} & 0 \\ 0 & d_{21} & -\bar{\gamma}_{21} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \gamma_{11} & d_{11} & 0 & 0 \\ d_{11} & -\bar{\gamma}_{11} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{42} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{32} & d_{32} \\ 0 & 0 & d_{32} & -\bar{\gamma}_{32} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{22} & d_{22} & 0 \\ 0 & d_{22} & -\bar{\gamma}_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{43} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{33} & d_{33} \\ 0 & 0 & d_{33} & -\bar{\gamma}_{33} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_{44} \end{bmatrix}$$

This is the product of ten matrices, six of which involve exactly one of the six parameters γ_{31}, γ_{21}, γ_{11}, γ_{32}, γ_{22}, and γ_{33}. For a unitary S, γ_{41}, γ_{42}, γ_{43}, and γ_{44} are modulus one constants. For an orthogonal S, γ_{41}, γ_{42}, γ_{43}, and γ_{44} are simply plus or minus signs.

In the following examples values are chosen for the parameters γ, and the resulting orthogonal matrices are shown.

$$\gamma_{41} = \gamma_{42} = \gamma_{43} = \gamma_{44} = 1$$
$$\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{22} = \gamma_{32} = \gamma_{33} = 0.05$$

$$S = \begin{bmatrix} 0.05 & 0.0499 & 0.0499 & 0.9963 \\ 0.0499 & 0.0498 & 0.996 & -0.0549 \\ 0.0499 & 0.996 & -0.055 & -0.0497 \\ 0.9963 & -0.0549 & -0.0497 & -0.0448 \end{bmatrix}$$

$$\gamma_{41} = \gamma_{42} = \gamma_{43} = \gamma_{44} = 1$$
$$\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{22} = \gamma_{32} = \gamma_{33} = \frac{1}{2}$$
$$S = \begin{bmatrix} 0.5 & 0.433 & 0.375 & 0.6495 \\ 0.433 & 0.25 & 0.433 & -0.75 \\ 0.375 & 0.433 & -0.8125 & -0.1083 \\ 0.6495 & -0.75 & -0.1083 & 0.0625 \end{bmatrix}$$

The above examples compute the orthogonal matrix S from the given parameters γ. Next the algorithm for determining the γ values for a given orthogonal matrix S is demonstrated.

We start with one of the matrices S above. The first set of parameters are found using the relationships shown below,

$$S = \begin{bmatrix} 0.5 & 0.4330 & 0.375 & 0.6495 \\ 0.4330 & 0.25 & 0.4330 & -0.75 \\ 0.3750 & 0.4330 & -0.8125 & -0.1083 \\ 0.6495 & -0.75 & -0.1083 & 0.0625 \end{bmatrix}$$

$$\gamma_{11} = s_{11} = 0.5 \qquad d_{11} = 0.866$$

$$\gamma_{21} = \frac{s_{21}}{d_{11}} = 0.5 \qquad d_{21} = 0.866$$

$$\gamma_{31} = \frac{s_{31}}{d_{21}d_{11}} = 0.5 \quad d_{31} = 0.866$$

$$\gamma_{41} = \frac{s_{41}}{d_{31}d_{21}d_{11}} = 1$$

These parameters are used to form the matrices

$$S_{11} = \begin{bmatrix} 0.5 & 0.866 & 0 & 0 \\ 0.866 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.866 \\ 0 & 0 & 0.866 & -0.5 \end{bmatrix} \qquad S_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the new matrix

$$S^1 = S_{11}{}^* S_{21}{}^* S_{31}{}^* S_{41}{}^* S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.433 & 0.75 \\ 0 & 0.433 & 0.625 & -0.6495 \\ 0 & 0.75 & -0.6495 & -0.125 \end{bmatrix}$$

From this new matrix, the next set of parameters are

$\gamma_{22} = s_{11} = 0.5 \qquad d_{22} = 0.866$

$\gamma_{32} = \dfrac{s_{32}}{d_{22}} = 0.5 \qquad d_{32} = 0.866$

$\gamma_{42} = \dfrac{s_{42}}{d_{32}d_{22}} = 1$

As before, we form the following matrices

$$S_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.866 \\ 0 & 0 & 0.866 & -0.5 \end{bmatrix} \quad S_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^2 = S_{22}{}^* S_{32}{}^* S_{42}{}^* S^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.866 \\ 0 & 0 & 0.866 & -0.5 \end{bmatrix}$$

Using matrix S^2, we find

$\gamma_{33} = s_{33}{}^2 = 0.5 \qquad d_{33} = 0.866$

$\gamma_{43} = \dfrac{s_{43}{}^2}{d_{33}} = 1$

and form

$$S_{33} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.866 \\ 0 & 0 & 0.866 & -0.5 \end{bmatrix} \qquad S_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^3 = S_{33}{}^* S_{43}{}^* S^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to get

$\gamma_{44} = s_{44}{}^3 = 1$

Returning to the general discussion, the transmission line realization for n = 4 is shown in Figure 4.

The effects of each of the four inputs on each of the four outputs can be seen by multiplication by the matrix S or through the transmission line representation equivalent. As an example, the dependence of the second output on the third input can be found by multiplying that input by the (3,3) entry of S_{33} times the (3,3) entry of S_{22} times the (3,3) entry of S_{32} times the (2,3) entry of S_{21} plus the (3,3) entry of S_{33} times the (2,3) entry of S_{22} times the (2,2) entry of S_{11} times the (2,2) entry of S_{21} plus the (3,4) entry of S_{33} times the modulus one constant γ_{43} times the (4,3) entry of S_{32} times the (2,3) entry of S_{21}. This process is related to the Dyck triangle and Catalan numbers in the Appendix.

Having shown the 2 x 2 and 4 x 4 cases, the general case is now shown. Let T_n be the family of triangular arrays.

$$A = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1,n-1} & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & & \gamma_{2,n-1} & \\ \vdots & & \iddots & & \\ \gamma_{n-1,1} & \gamma_{n-1,2} & & & \\ \gamma_{n1} & & & & \end{bmatrix} \tag{28}$$

such that

- $\left|\gamma_{ij}\right| \le 1$
- $\left|\gamma_{k,n-k+1}\right| = 1$ for $k = 1, \ldots, n$
- if $\left|\gamma_{ij}\right| = 1$ for some $1 \le i \le n-1$ and $1 \le j \le n+1-i$, then $\gamma_{i,j+k} = 0$ for $1 \le k < n+1-i-j$ and $\gamma_{i,n+1-i} = 1$

<u>Theorem 1</u> There exists a one-to-one mapping Φ from T_n onto $U(n)$, such that

$$\Phi(A)\left(=U(A)\right)=\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&\gamma_{n1}\end{bmatrix}\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&J(\gamma_{n-1,1})\end{bmatrix}\cdots\begin{bmatrix}J(\gamma_{11})&&&\\&1&&\\&&\ddots&\\&&&1\end{bmatrix}$$

$$\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&\gamma_{n-1,2}\end{bmatrix}\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&J(\gamma_{n-2,2})\end{bmatrix}\cdots\begin{bmatrix}1&&&\\&J(\gamma_{1,2})&&\\&&\ddots&\\&&&1\end{bmatrix}$$

$$\vdots$$

$$\cdots\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&\gamma_{2,n-1}\end{bmatrix}\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&J(\gamma_{1,n-1})\end{bmatrix}\begin{bmatrix}1&&&\\&\ddots&&\\&&1&\\&&&\gamma_{1n}\end{bmatrix}$$

Proof: The above proof for n = 2 and n = 4 works as well in this general case.

The general transmission line is shown in Figure 5.

Decomposing unitary matrices is an aid in identifying system parameters experimentally. Given a four-port coupler the corresponding sixteen parameters satisfying the ten constraints can be found from input/output experiments. If done directly, sixteen input/output pairs are needed. However, the six parameters γ_{11}, γ_{21}, γ_{31}, γ_{22}, γ_{32}, and γ_{33} and the modulus one constants γ_{41}, γ_{42}, γ_{43}, and γ_{44} can be identified from ten input/output pairs, from which the orthogonal matrix S representing the coupler can be constructed.

A procedure using the transmission line approach for n = 4 computes the parameters in the order indicated by our algorithm. In each step the parameter or modulus one constant is found by solving one linear equation with one unknown (assuming the coefficient is nonzero):

From input 1 to output 1, γ_{11} (and hence S_{11}) is found.

From input 1 to output 2, γ_{21} (and hence S_{21}) is found.

From input 1 to output 3, γ_{31} (and hence S_{31}) is found.

From input 1 to output 4, γ_{41} is found.

From input 2 to output 2, γ_{22} (and hence S_{22}) is found.

From input 2 to output 3, γ_{32} (and hence S_{32}) is found.

From input 2 to output 4, γ_{42} is found.

From input 3 to output 3, γ_{33} (and hence S_{33}) is found.

From input 3 to output 4, γ_{43} is found.

From input 4 to output 4, γ_{44} is found.

At each step in the algorithm for computing the γ parameters from the unitary matrix S there are assumptions that are intrinsically being made. In our experimental method given above these assumptions result in the fact that each γ is found by solving one linear equation with one unknown where the coefficient is nonzero. An example of an intrinsic assumption in the $n = 2$ case is that the modulus of γ_{11} must be assumed to be less than one in order to have that γ_{21} is uniquely determined by S.

Examples illustrating what happens if the intrinsic assumptions do not hold are now provided. For the case $n = 2$ consider the orthogonal matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in which $\gamma_{11} = 1$ and $d_{11} = 0$. From the transmission line description (Figure 3), $\gamma_{21} = 1$ and $\gamma_{12} = -1$ or $\gamma_{21} = -1$ and $\gamma_{12} = 1$, and the solution is no longer unique.

For an example when $n = 4$, consider the orthogonal matrix

$$S = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\gamma_{11} = s_{11} = -0.5 \qquad d_{11} = 0.866$$

$$\gamma_{21} = \frac{s_{21}}{d_{11}} = .5774 \qquad d_{21} = 0.8165$$

$$\gamma_{31} = \frac{s_{31}}{d_{21}d_{11}} = 0.7071 \quad d_{31} = 0.7071$$

$$\gamma_{41} = \frac{s_{41}}{d_{31}d_{21}d_{11}} = 1$$

These parameters are used to form the new matrix

$$S^1 = S_{11}{}^* S_{21}{}^* S_{31}{}^* S_{41}{}^* S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5774 & 0.5774 & 0.5774 \\ 0 & -0.8165 & 0.4083 & 0.4083 \\ 0 & 0 & -0.7071 & 0.7071 \end{bmatrix}$$

From this new matrix, the next set of parameters are

$$\gamma_{22} = s_{22}{}^1 = 0.5774 \qquad d_{22} = 0.8165$$

$$\gamma_{32} = \frac{s_{32}{}^1}{d_{22}} = -1 \quad d_{32} = 0$$

At this point the algorithm fails as $|\gamma_{32}| = 1$. The transmission line description determines γ_{33} by considering the relationship between input 3 and output 1:

$$\gamma_{33} d_{22} d_{11} = \frac{1}{2} \text{ yielding } \gamma_{33} = 0.7071 \text{ and } d_{33} = 0.7071$$

Then γ_{44} is determined by considering the relationship between input 4 and output 1:

$$\gamma_{44} d_{33} d_{22} d_{11} = \frac{1}{2} \text{ yielding } \gamma_{44} = 1$$

Finally, considering the relationship between input 4 and output 4 yields

$$\gamma_{41}d_{31}\gamma_{21}\gamma_{32}\gamma_{22}d_{33}\gamma_{44}-\gamma_{41}d_{31}d_{21}\gamma_{11}d_{22}d_{33}\gamma_{44}-\gamma_{41}\gamma_{31}\gamma_{42}\gamma_{32}\gamma_{43}\gamma_{33}\gamma_{44}=\frac{1}{2}$$

or, equivalently,

$$\frac{\gamma_{42}\gamma_{43}}{2}=-\frac{1}{2}$$

Therefore $\gamma_{42} = 1$ and $\gamma_{43} = -1$ or $\gamma_{42} = -1$ and $\gamma_{43} = 1$, which again is not a unique solution since there are two sets of γ_{42} and γ_{43} that give the same orthogonal matrix.

4. Conclusions

In this paper, two related approaches are provided to determine the elements of the S-matrix for a four-port coupler. Examples provide context for these mathematical procedures. The Cayley map represents the S-matrix in terms of a skew symmetric matrix and allows one to transform between the sixteen elements of the S-matrix and the six free parameters that specify it, and vice-versa. The second method provides an algorithmic decomposition of the S-matrix, and has a physical interpretation in terms of coupled transmission lines. This latter method is related to the theory and application of reduced port network analyzers [27, 28].

This work has direct relevance to the researchers work in developing design techniques for filters based on the 4-port couplers discussed in this paper [11]. Part of the algorithm for filter realization sets the desired coupler coefficients. The techniques described here are useful in generating couplers that are realizable.

Although this work is limited to four-port couplers, both approaches can be scaled to higher order S-matrices and more general couplers. For example, the researchers are particularly interested in compact couplers that may be described by real coefficients, and the examples considered here reflect this. The techniques presented here, however, are directly applicable to S-matrices with complex elements. This description is particularly useful to model the transit delay through the device, and often this delay can be included explicitly by left- and right- multiplying S by diagonal phase matrices, to form $S' = \Phi S \Psi$ [35].

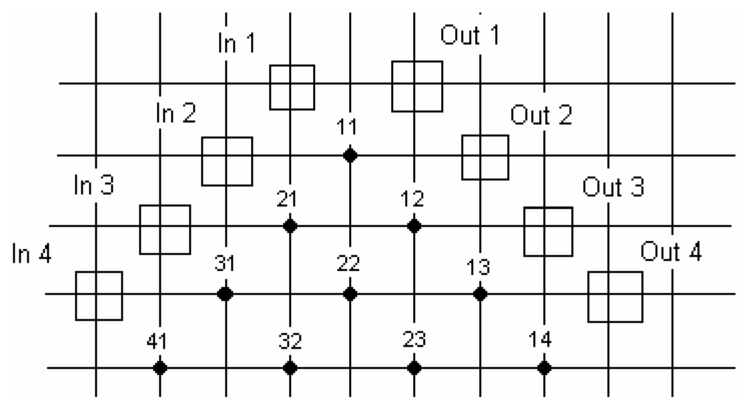
As photonics continues to grow in integration, photonic circuits will become more sophisticated. As in microwaves, four-port and higher couplers are emerging as useful building blocks. This paper adds to the existing general literature on the four-port coupler in the context of photonics, and in particular the context of higher order optical lattice filters. This perspective has provided certain advantages that allow advances over the previous work and can be useful in other fields of engineering and physics.

Appendix A: The Combinatorial Structure

In terms of the parameterization of Theorem 1 for the case $n = 2$, the entries of G can be explicitly denoted

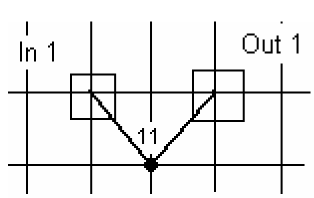
$$S = \begin{bmatrix} \gamma_{11} & d_{11}\gamma_{12} \\ \gamma_{21}d_{11} & -\gamma_{21}\bar{\gamma}_{11}\gamma_{12} \end{bmatrix} \tag{A1}$$

A combinatorial description of the entries of S for arbitrary n is now introduced. Consider the following configuration in $\mathbb{Z}^2$ (Cartesian product of two copies of the integers):

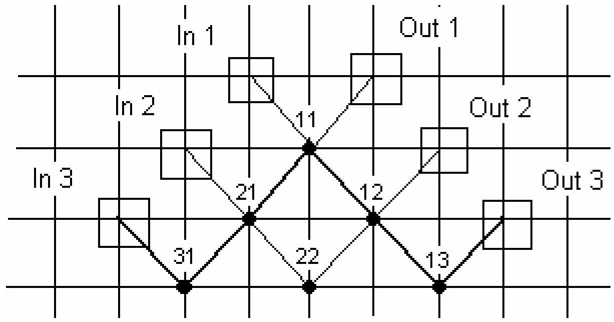


The black dots are the admissible points and their set is denoted B_n. We consider F_{ij}^n to be the set of paths through admissible points from i to j consisting of only two types of steps: falls and rises.

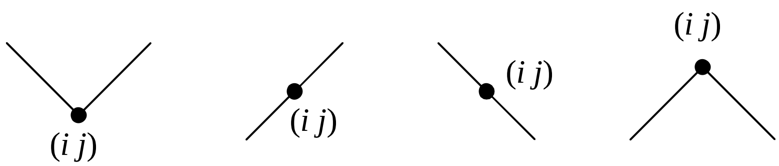
For instance F_{11}^2 consists of only one path:



A more complex path is illustrated for n = 3:



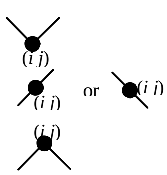
Notice that a path has only four types of behavior when crossing a point $(i, j) \in B_n$, namely



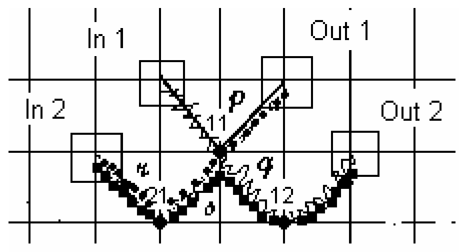
Let $A \in T_n$. The following functions are considered:

$$a_{ij} : \bigcup_{1 \le k, l \le n} S^{n}_{k_1 l}$$

$$a_{ij}(p) = \begin{cases} \gamma_{ij} & \text{if p at } (i, j) \text{ is like} \\ d_{ij} & \text{if p at } (i, j) \text{ is like} \\ -\bar{\gamma}_{ij} & \text{if p at } (i, j) \text{ is like} \\ 1 & \text{if } (i, j) \notin p \end{cases} \qquad \text{(A2)}$$



Now, returning to the case n = 2, notice that (this diagram should be compared with our description of the transmission line in the case n = 2):



$$s_{11} = a_{11}(p) = \sum_{p \in S_{11}^2} \prod_{(i,j) \in B_2} a_{ij}(p) \tag{A3}$$

$$s_{12} = d_{11}\gamma_{12} = a_{11}(Q)a_{12}(Q) = \sum_{p \in S_{12}^2} \prod_{(i,j) \in B_2} a_{ij}(p) \tag{A4}$$

$$s_{21} = \gamma_{21}d_{11} = a_{21}(\gamma)a_{11}(\gamma) = \sum_{p \in S_{21}^2} \prod_{(i,j) \in B_2} a_{ij}(p) \tag{A5}$$

$$s_{22} = -\gamma_{21}\bar{\gamma}_{11}\gamma_{12} = a_{21}(s)a_{11}(s)a_{12}(s) = \sum_{p \in S_{22}^2} \prod_{(i,j) \in B_2} a_{ij}(p) \tag{A6}$$

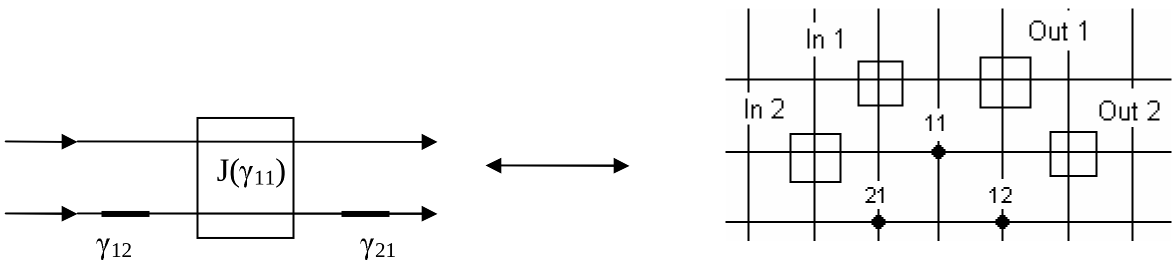
The same structure holds in general:

Theorem 2 Let $S \in U(n)$, $U = \Phi(A)$ for some $A \in T_n$. Then

$$s_{ij}(S) = \sum_{p \in S_{ij}^n} \prod_{(ij) \in B_n} a_{ij}(p) \tag{A7}$$

where s_{ij} is the (i, j) entry of G.

Proof The main point of the proof consists in identifying a box in the transmission line representation of S with an element of B_n. For instance:



Interestingly enough the sets of S_{ij} are related to the paths enumerated by the Dyck triangle [31] which is a classical object in combinatorics.

More precisely, the Dyck triangle counts the paths in the positive quadrant of $\mathbb{Z}^2$ starting at origin and consisting of fall and rise steps. Let c_{ij} be the number of such paths ending at (i, j). Clearly $c_{ij} \neq 0$ if and only if $i \geq j$ and $i + j$ is even. Omitting the 0 entries, the Dyck triangle is given by

1

1 5

1 4 14

1 3 9 28

1 2 5 14 42

$\underline{1}$ $\underline{1}$ $\underline{2}$ $\underline{5}$ $\underline{14}$ $\underline{42}$

The underlined numbers are the Catalan numbers.

Theorem For $i \le j$,

$$\begin{aligned} &\#\left(S_{ij}^{n}\right) = c_{j+i-1,\, j-i+1}, \qquad 1 \le i \le j \le n \\ &\#\left(S_{ij}^{n}\right) = c_{j+i-2,\, n-i}, \qquad 1 \le i \le n \\ &\#\left(S_{ij}^{n}\right) = \#\left(S_{ji}^{n}\right) \end{aligned} \tag{A8}$$

where $\#\left(S_{ij}^{n}\right)$ denotes the number of elements of the set S_{ij}^{n} .

Proof By simple inspection.

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Figure Captions

Figure 1. Two deep trenches perpendicular to each other and oriented at 45º to 3 um-wide ridge waveguide. The four-port coupler is milled with Focused Ion Beam.

Figure 2. Signal flow diagram of a 4-direction coupler that reflects, transmits, routes left and routes right. In our notation, the 4 ports are labeled W, N, E and S. At each port, there is a reflected component, ρ, a transmitted component, τ, a right directed component, α, and a left directed component, β. Shown explicitly in this diagram are ρw, τw, αw, and βw, the coupling coefficients at the W port.

Figure 3. The transmission line interpretation of the matrix multiplication of a 2x2 decomposition into simple unitary matrices.

Figure 4. The transmission line interpretation of the matrix multiplication of a 4x4 decomposition into simple unitary matrices.

Figure 5. The general transmission line interpretation of the matrix multiplication of an nxn decomposition into simple unitary matrices.

Figures

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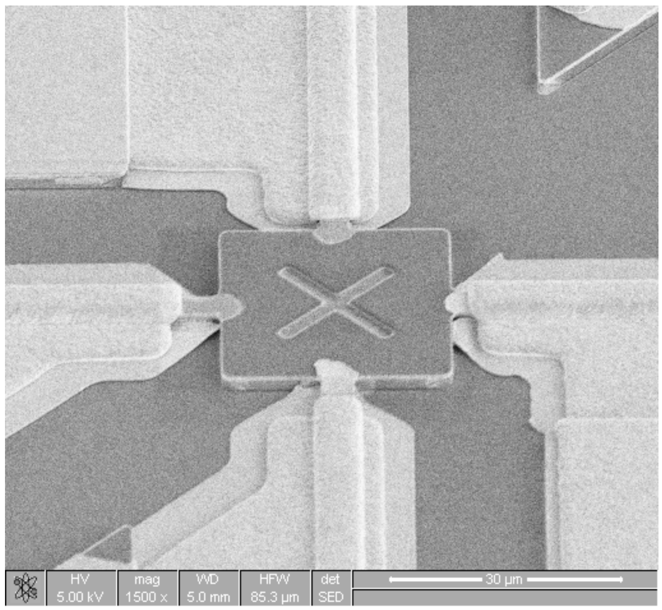

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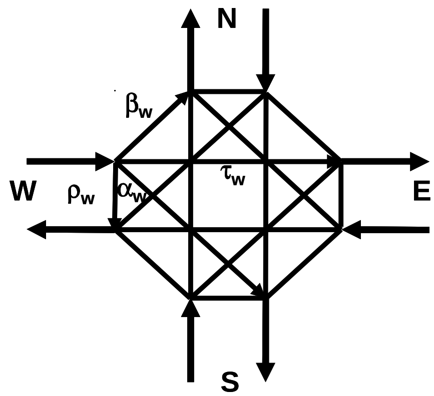

Figure 3. The transmission line interpretation of the matrix multiplication of a 2x2 decomposition into simple unitary matrices.

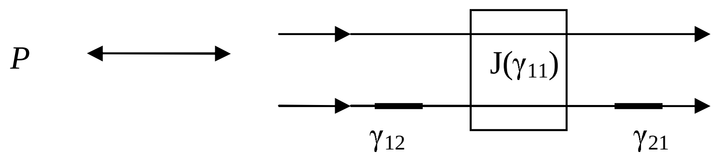

Figure 4. The transmission line interpretation of the matrix multiplication of a 4x4 decomposition into simple unitary matrices.

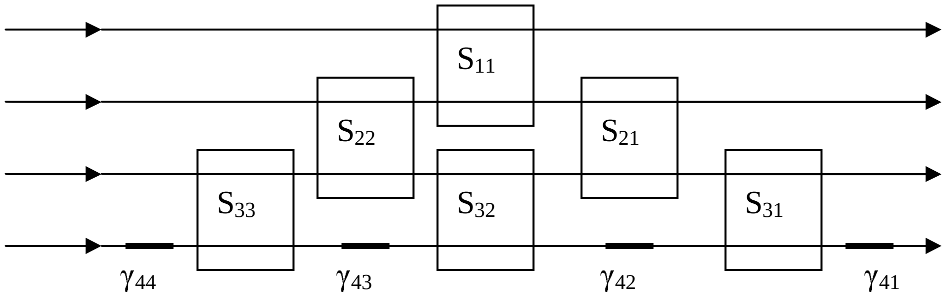

Figure 5. The general transmission line interpretation of the matrix multiplication of an nxn decomposition into simple unitary matrices.