A mathematical analysis of the frequency response of the wavefront coding odd-symmetric quadratic phase mask is presented. An exact solution for the optical transfer function of a wavefront coding imager using this type of mask is derived from first principles, whose result applies over all misfocus values. The misfocus dependent spatial filtering property of this imager is described. The available spatial frequency bandwidth for a given misfocus condition is quantified. A special imaging condition that yields an increased dynamic range is identified. © 2006 Optical Society of America

OCIS codes: 110.4850, 070.6110.

1. Introduction
In recent years, researchers have developed a computational imaging technique called wavefront coding to create imaging systems that are capable of achieving extended depths of field. Wavefront coding enables these systems to operate over an extended depth of field by modifying the light field at the aperture of these optical systems so as to permit imaging even in the presence of considerable misfocus. In the original version of this technique as it was introduced in Ref 1, a cubic phase modulation (cubic-pm) mask is placed at the exit pupil of a conventional imager, thereby transforming the spatial frequency response of the imager. The optical transfer function (OTF) of such a system varies little in magnitude with misfocus and contains no nulls within its passband. The optically formed image captured by the detector therefore exhibits a uniform blur which is largely independent of misfocus. This intermediate image can then be digitally processed with a simple linear restoration filter to yield a final image with greatly improved depth of focus. Figure 1 describes one such wavefront coding system.

Once wavefront coding with the cubic phase mask demonstrated that enhanced imaging performance could be achieved by deliberately blurring an image in a calculated way, researchers began to look for methods to optimize the nature of the blur in order to maximize the quality of the post-processed image. Various image quality metrics were subsequently proposed to measure imager performance and tailor the pupil phase profile in order to obtain even better results than the cubic phase mask. Wavefront coding was also extended to circular and annular pupils, and numerous circular phase profiles were studied. Even though circular apertures are a staple of a majority of imagers, rectangularly separable systems retain the advantage of faster image reconstruction due to the reduced computational overhead associated with the separate signal processing along each of the image plane dimensions. Moreover, the ambiguity functions (AF) of rectangularly separable systems may be treated in just two dimensions, a feature available to phase profiles with circular apertures only if they exhibit radial symmetry.

The appeal of wavefront coding as a new paradigm for optical system design lies in the elegance of its simplicity and its ability to address multiple issues simultaneously. It has been demonstrated that wavefront coding techniques reduce complexity in optical design and is capable of correcting or minimizing the effects of a host of aberrations such as Petzval curvature, astigmatism, chromatic aberration, spherical
aberration and temperature-related misfocus.\textsuperscript{13-17} In this work, the behavior of a wavefront coding imager incorporating a phase mask with an odd-symmetric quadratic phase profile is examined by conducting a mathematical analysis of its spatial frequency response. Development of an exact analytical representation of the OTF of a cubic-pm system allowed a comprehensive spatial frequency analysis of such an imager.\textsuperscript{18} Knowledge of the exact OTF of a cubic-pm imager helped quantify its available spatial frequency bandwidth and enabled the system design and tradeoff analysis of this system.

In this work, a similar mathematical analysis on the frequency response of a wavefront coding imager with an odd-symmetric quadratic phase modulation element is performed. An exact representation of the OTF of such a system is presented, and the available spatial frequency bandwidth and special imaging conditions that yield an enhanced dynamic range of this imager are described. The improved noise handling ability due to this enhanced dynamic range could make this odd-symmetric quadratic phase modulation element an attractive alternative to the cubic phase mask in applications where noise issues dominate over other system performance requirements.

2. The Odd-Symmetric Quadratic Phase Mask

Research on rectangularly separable systems has shown that a phase plate that extends the depth of field must have an odd-symmetric phase profile;\textsuperscript{19} that is, the phase function must satisfy the condition

\begin{equation}
\eta(x, y) = -\eta(-x, -y).
\end{equation}

The above condition implies that a phase plate that yields an extended depth of focus must itself lack any focusing power. The results have been used to derive phase plates with a logarithmic contour that have similar properties as those of the cubic phase mask.\textsuperscript{19} A family of phase masks that have also been proposed as good candidates for enhancing the depth of field are described along one dimension (1-D) by the phase function,\textsuperscript{15,20,21}

\begin{equation}
\eta(x) = \alpha \text{sign}(x)|x|^\gamma ; \ -1 \leq x \leq 1, \ \alpha > 0, \ \gamma \geq 2.
\end{equation}

In the above equation, $x$ is the normalized pupil plane coordinate, $\alpha$ is a positive design constant that controls the phase deviation and hence the strength of the phase mask, and $\gamma$ is a positive real power. The signum function in Eq. (2) contributes to the odd symmetry of the phase profile and is given by

\begin{equation}
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x < 0.
\end{cases}
\end{equation}
The properties of imaging systems with fractional values of $\gamma$ have been studied, and numerical investigations into the behavior of the modulation transfer function (MTF) of a mask with $\gamma = 2$ have been conducted.

In this work, the OTF of a member of the family of phase masks described by Eq. (2) is mathematically evaluated. Specifically, an analytical expression for the OTF for a phase mask with $\gamma = 2$ is derived and the result is exploited to evaluate the available spatial frequency bandwidth of this imaging system for a given value of misfocus. This phase mask is herein termed the odd-symmetric quadratic (OSQ) phase mask. This work also identifies a special imaging condition that yields an increased dynamic range of the imager.

A. Phase Mask Partitioning

The phase profile of Eq. (2) for $\gamma = 2$ may be expressed in conjunction with the definition of the signum function given by Eq. (3) as

$$\eta(x) = \begin{cases} -ax^2 & ; -1 \leq x < 0 \\ ax^2 & ; 0 \leq x \leq 1 \end{cases}.$$

The generalized pupil function of the wavefront coding imager with the above phase profile is then given by

$$P(x) = \begin{cases} \frac{1}{\sqrt{2}} \exp\left[ j(\psi-a)x^2 \right] = P_-(x) ; & -1 \leq x < 0 \\ \frac{1}{\sqrt{2}} \exp\left[ j(\psi+a)x^2 \right] = P_+(x) ; & 0 \leq x \leq 1 \\ 0 ; & \text{otherwise} \end{cases}.$$

Here, $\psi$ is the misfocus parameter and is defined as,

$$\psi = \frac{\pi L^2}{4\lambda} \left( \frac{1}{f} - \frac{1}{d_o} - \frac{1}{d_i} \right).$$

B. OTF of the Odd-Symmetric Quadratic Phase Mask System

Following the techniques developed in the analytical evaluation of the OTF of a cubic-pm imager, the OTF of a rectangularly separable OSQ phase mask system along one dimension may similarly be calculated from first principles by analytically evaluating the autocorrelation of the generalized pupil function. Evaluating the OTF of the imager whose pupil function is given in Eq. (5) from first principles requires careful consideration of the area of overlap in the autocorrelation integral due to the partitioning of the pupil function. For a general phase mask $P(x)$ that is partitioned into two sections $P_-(x)$ and $P_+(x)$ about $x = 0$, the autocorrelation process may be split into four distinct regions and the OTF written as

$$H(u,\psi) = \int_{-1}^{1} P(x+u)P^*(x-\psi)du \quad ; \quad -1 \leq u \leq 1$$

$$+ \int_{-1/2}^{-1} P(x+u)P^*(x-\psi)du \quad ; \quad -1 \leq u \leq 0$$

$$+ \int_{1}^{1/2} P(x+u)P^*(x-\psi)du \quad ; \quad 0 \leq u \leq 1$$

Incorporating the actual values of $P_-(x)$ and $P_+(x)$ from Eq. (5) into Eq. (7), the OTF of the OSQ phase mask system may be expressed as
Eq. (8) may be further simplified by exploiting the symmetry of the kernel of the integrals about \( u = 0 \). The OTF may therefore be restated as

\[
H(u, \psi) = \begin{cases} 
\frac{1}{2} \int_{-u}^{1} \exp \left[ j \left( 4u\psi - 2\alpha(x^2 + u^2) \right) \right] dx & ; \ -1 \leq u \leq -\frac{1}{2} \\
\frac{1}{2} \int_{-u}^{1} \exp \left[ j4u(\psi - \alpha) x \right] dx + \frac{1}{2} \int_{u}^{1} \exp \left[ j \left( 4u\psi - 2\alpha(x^2 + u^2) \right) \right] dx & ; \ -\frac{1}{2} \leq u \leq 0 \\
\frac{1}{2} \int_{-u}^{1} \exp \left[ j4u(\psi - \alpha) x \right] dx + \frac{1}{2} \int_{u}^{1} \exp \left[ j \left( 4u\psi + 2\alpha(x^2 + u^2) \right) \right] dx & ; \ 0 \leq u \leq \frac{1}{2} \\
\frac{1}{2} \int_{-u}^{1} \exp \left[ j4u(\psi + \alpha) x \right] dx + \frac{1}{2} \int_{u}^{1} \exp \left[ j \left( 4u\psi + 2\alpha(x^2 + u^2) \right) \right] dx & ; \ \frac{1}{2} \leq u \leq 1
\end{cases}
\] (8)

For ease of notation, the OTFs within the two distinct regions in Eq. (9) are referred to as the center \( H_C(u, \psi) \) and tails \( H_T(u, \psi) \) such that

\[
H(u, \psi) = \begin{cases} 
\frac{1}{2} \int_{-|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx + \frac{1}{2} \int_{|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx & ; \ 0 \leq |u| \leq \frac{1}{2} \\
\frac{1}{2} \int_{-|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx + \frac{1}{2} \int_{|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx & ; \ \frac{1}{2} \leq |u| \leq 1
\end{cases}
\] (9)

\[
H_C(u, \psi) = \frac{1}{2} \int_{-|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx + \frac{1}{2} \int_{|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx \\
\quad + \frac{1}{2} \int_{-|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx + \frac{1}{2} \int_{|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx & ; \ 0 \leq |u| \leq \frac{1}{2}
\] (10)

and

\[
H_T(u, \psi) = \frac{1}{2} \int_{-|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx + \frac{1}{2} \int_{|u|}^{1} \exp \left[ j4|\psi|\alpha x \right] dx & ; \ \frac{1}{2} \leq |u| \leq 1.
\] (11)
The tails of the OTF are evaluated first. The right-hand side of the above equation may be rewritten after completing the square in the exponent as

\[ H_T(u, \psi) = \frac{1}{2} \exp \left[ j2au^2 \left( 1 - \frac{\psi^2}{\alpha^2} \right) \text{sign}(u) \right] \]

\[ \times \int_{b_T(u)}^{a_T(u)} \exp \left[ j \frac{\pi}{2} \left( \frac{4\alpha}{\pi} \right) x + \frac{\psi}{\alpha} |u| \right] \text{sign}(u) \right] dx \quad \frac{1}{2} \leq |u| \leq 1. \]

Applying a change of variables on the integral, the above equation may be expressed with the new limits as

\[ H_T(u, \psi) = \frac{\left( \frac{\pi}{4a} \right)^{1/2}}{\left( \frac{\pi}{4a} \right)} \times \exp \left[ j2au^2 \left( 1 - \frac{\psi^2}{\alpha^2} \right) \text{sign}(u) \right] \]

\[ \times \int_{b_T(u)}^{a_T(u)} \exp \left[ j \frac{\pi}{2} \tau^2 \text{sign}(u) \right] \right] \frac{1}{2} \leq |u| \leq 1 \]

where the integration limits \( a_T(u) \) and \( b_T(u) \) are given by the relation

\[ a_T(u) = \left( \frac{4\alpha}{\pi} \right) \left[ |u| \left( 1 + \frac{\psi}{\alpha} \right) - 1 \right] \]

\[ b_T(u) = \left( \frac{4\alpha}{\pi} \right) \left[ 1 - |u| \left( 1 + \frac{\psi}{\alpha} \right) \right] \]

The integrals are identified as Fresnel cosines and Fresnel sines. The OTF at the tails may then be expressed as

\[ H_T(u, \psi) = \left( \frac{\pi}{8a} \right)^{1/2} \exp \left[ j2au^2 \left( 1 - \frac{\psi^2}{\alpha^2} \right) \text{sign}(u) \right] \]

\[ \times \int_{b_T(u)}^{a_T(u)} \exp \left[ j \frac{\pi}{2} \tau^2 \text{sign}(u) \right] \right] \frac{1}{2} \leq |u| \leq 1 \]

The operators \( C(\cdot) \) and \( S(\cdot) \) represent the Fresnel cosine integral and Fresnel sine integral respectively. In order to evaluate the central portion of the OTF, the term \( H_C(u, \psi) \) is further split into its three constituent sections, each section representing one of the integration expressions. Therefore, \( H_C(u, \psi) = I_1(u, \psi) + I_2(u, \psi) + I_3(u, \psi) \), where

\[ I_1(u, \psi) = \frac{1}{2} \int_{-|u|}^{|u|} \exp \left[ j4u(\psi - \alpha)x \right] \right] \]

\[ I_2(u, \psi) = \frac{1}{2} \int_{-|u|}^{|u|} \exp \left[ j4u|x| \right] \right] \]

\[ I_3(u, \psi) = \frac{1}{2} \int_{-|u|}^{|u|} \exp \left[ j4u(\psi + \alpha)x \right] \right] \]

These three terms are evaluated separately and then combined to form the OTF at the central portion of the spatial frequency range. Performing the integration operation on \( I_1(u, \psi) \) results in
\[ I_1(u, \psi) = \frac{1}{2} \int_{-\infty}^{\infty} \exp\left[-j4u(\psi - \alpha)x\right] dx \]
\[ = \frac{1}{2} \exp\left[-j4u(\psi - \alpha)\right] \int_{-\infty}^{\infty} \exp\left[j2u(\psi - \alpha)x\right] dx. \]  

Let \( \tau_{C1} = \frac{1}{2} - |u| \) so that \(-|u| = \tau_{C1} - \frac{1}{2} \) and \(|u| - 1 = -\tau_{C1} - \frac{1}{2} \). Eq. (18) then becomes

\[ I_1(u, \psi) = \frac{1}{2j4u(\psi - \alpha)} \times \left\{ \exp\left[j4u(\psi - \alpha)\right] \left(\tau_{C1} - \frac{1}{2}\right) - \exp\left[-j4u(\psi - \alpha)\right] \left(-\tau_{C1} - \frac{1}{2}\right) \right\}, \]  

which may be rewritten as

\[ I_1(u, \psi) = \frac{\exp\left[-j2u(\psi - \alpha)\right]}{4u(\psi - \alpha)} \times \left\{ \frac{\exp\left[j4u(\psi - \alpha)\right] \tau_{C1}}{2j} - \frac{\exp\left[-j4u(\psi - \alpha)\right] \tau_{C1}}{2j} \right\}. \]  

Euler’s identity is once again invoked on the term within the curly parentheses to obtain

\[ I_1(u, \psi) = \frac{\sin\left[4u(\psi - \alpha)\right]}{4u(\psi - \alpha)} \times \left\{ \frac{\sin\left[4u(\psi - \alpha)\right] \tau_{C1}}{4u(\psi - \alpha)} \right\}. \]  

By first multiplying the numerator and denominator of Eq. (21) by \( \tau_{C1} \), then multiplying and dividing the denominator as well as the argument of the sine function by \( \pi \), and finally reinstating the value of \( \tau_{C1} \), the term \( I_1(u, \psi) \) is obtained as

\[ I_1(u, \psi) = \left(\frac{1}{2} - |u|\right) \sin\left[\frac{4u}{\pi} (\psi - \alpha) \left(\frac{1}{2} - |u|\right)\right] \times \exp\left[-j2u(\psi - \alpha)\right]. \]  

Next, the term \( I_2(u, \psi) \) is calculated. This term is similar to the integral encountered in the analysis of \( H_1(u, \psi) \) except for the limits of the integral. Therefore, it is possible to write

\[ I_2(u, \psi) = \frac{1}{2} \exp\left[-j2au^2\left(1 - \frac{\psi^2}{\alpha^2}\right)\right] \times \int_{-\infty}^{\infty} \exp\left[j\frac{\pi}{2} \left(\frac{4\alpha}{\pi}\right) \left(x + \frac{\psi}{\alpha}u\right)^2\right] \sin(u) dx. \]  

Application of a change of variable in the integral of the above equation allows this expression to be restated as

\[ I_2(u, \psi) = \frac{1}{4} \left(\frac{\pi}{\alpha}\right)^{1/2} \times \exp\left[-j2au^2\left(1 - \frac{\psi^2}{\alpha^2}\right)\right] \times \int_{a_C(u)}^{b_C(u)} \exp\left[j\frac{\pi}{2} \tau_c^2 \sin(u)\right] d\tau_c, \]  

where the integration limits \( a_C(u) \) and \( b_C(u) \) are given by the relation

\[ a_C(u) = \left(\frac{4\alpha}{\pi}\right)^{1/2} \left|\frac{\psi}{\alpha} - 1\right|; \]
\[ b_C(u) = \left(\frac{4\alpha}{\pi}\right)^{1/2} \left|\frac{\psi}{\alpha} + 1\right|. \]  

As in the case of \( H_1(u, \psi) \), Euler’s identity is applied on the kernel to yield

\[ I_2(u, \psi) = \frac{1}{4} \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left[-j2au^2\left(1 - \frac{\psi^2}{\alpha^2}\right)\right] \times \int_{a_C(u)}^{b_C(u)} \cos\left(\frac{\pi}{2} \tau_c^2\right) d\tau_c + j \sin(u) \int_{a_C(u)}^{b_C(u)} \sin\left(\frac{\pi}{2} \tau_c^2\right) d\tau_c. \]  

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The integrals in the above equation are then expressed as Fresnel cosines and Fresnel sines. The resulting expression for \( I_2(u, \psi) \) is then

\[
I_2(u, \psi) = \left( \frac{\pi}{8\alpha} \right)^{1/2} \exp \left[ j2\alpha u^2 \left( 1 - \frac{\psi^2}{\alpha^2} \right) \right] \text{sign}(u) \times \frac{1}{\sqrt{2}} \left[ C(b_c(u)) - C(a_c(u)) \right] + j \text{sign}(u) \left[ S(b_c(u)) - S(a_c(u)) \right].
\] (27)

The term \( I_3(u, \psi) \) is alike in nature to \( I_1(u, \psi) \) and may be evaluated similarly. Performing the integration on \( I_3(u, \psi) \) results in

\[
I_3(u, \psi) = \frac{1}{2} \int_{|u|}^{1-|u|} \exp \left[ j4u(\psi + \alpha) x \right] dx
= \frac{1}{2} \exp \left[ j4u(\psi + \alpha) \right] \left[ 1 - \exp \left[ j4u(\psi + \alpha)(\tau_{c3} + \frac{1}{2}) \right] - \exp \left[ j4u(\psi + \alpha)(-\tau_{c3} + \frac{1}{2}) \right] \right].
\] (28)

Let \( \tau_{c3} = \frac{1}{2} - |u| \) so that \( 1 - |u| = \tau_{c3} + \frac{1}{2} \) and \( |u| = -\tau_{c3} + \frac{1}{2} \). Eq. (28) then becomes

\[
I_3(u, \psi) = \frac{1}{2} j 4u(\psi + \alpha) \times \left\{ \exp \left[ j4u(\psi + \alpha)(\tau_{c3} + \frac{1}{2}) \right] - \exp \left[ j4u(\psi + \alpha)(-\tau_{c3} + \frac{1}{2}) \right] \right\}.
\] (29)

which may be expressed as

\[
I_3(u, \psi) = \frac{\exp \left[ j2u(\psi + \alpha) \right]}{4u(\psi + \alpha)} \times \left\{ \frac{\exp \left[ j4u(\psi + \alpha) \tau_{c3} \right]}{2j} - \frac{\exp \left[ -j4u(\psi + \alpha) \tau_{c3} \right]}{2j} \right\}.
\] (30)

Utilizing Euler’s identity on the term within the curly parentheses yields

\[
I_3(u, \psi) = \exp \left[ j2u(\psi + \alpha) \right] \times \left\{ \sin \left[ 4u(\psi + \alpha) \tau_{c3} \right] \right\}.
\] (31)

The above equation may be expressed in terms of a sinc function similar to that seen in Eq. (22). Straightforward algebraic manipulation produces

\[
I_3(u, \psi) = \left( \frac{1}{2} - |u| \right) \text{sinc} \left[ \frac{4u}{\pi} \left( \psi + \alpha \right) \left( \frac{1}{2} - |u| \right) \right] \times \exp \left[ j2u(\psi + \alpha) \right].
\] (32)

Combining the results for \( I_1(u, \psi) \), \( I_2(u, \psi) \) and \( I_3(u, \psi) \) gives the expression for \( H_c(u, \psi) \) as

\[
H_c(u, \psi) = \left( \frac{1}{2} - |u| \right) \text{sinc} \left[ \frac{4u}{\pi} \left( \psi - \alpha \right) \left( \frac{1}{2} - |u| \right) \right] \exp \left[ -j2u(\psi - \alpha) \right] \frac{\pi}{8\alpha}^{1/2} \exp \left[ j2\alpha u^2 \left( 1 - \frac{\psi^2}{\alpha^2} \right) \right] \text{sign}(u) \times \frac{1}{\sqrt{2}} \left[ C(b_c(u)) - C(a_c(u)) \right] + j \text{sign}(u) \left[ S(b_c(u)) - S(a_c(u)) \right] + \left( \frac{1}{2} - |u| \right) \text{sinc} \left[ \frac{4u}{\pi} (\psi + \alpha) \left( \frac{1}{2} - |u| \right) \right] \exp \left[ j2u(\psi + \alpha) \right] ; \ 0 \leq |u| \leq \frac{1}{2}.
\] (33)
Eqs. (16), (22), (27) and (32) together form the OTF of the odd-symmetric quadratic phase mask. The magnitude of this OTF at \( u = 0 \) is obtained by evaluating \( H_c(u, \psi) \) at the zero-frequency location. Direct inspection of Eqs. (22) and (32) at \( u = 0 \) reveals that \( I_1(0, \psi) = I_3(0, \psi) = \frac{1}{2} \). Similarly, inspecting Eq. (25) shows that \( a_c(0) = b_c(0) = 0 \), which when incorporated into Eq. (27) indicates that \( I_2(0, \psi) = 0 \). From these results, it is seen that

\[
H_c(0, \psi) = 1. \tag{34}
\]

Therefore, the normalized OTF of the odd-symmetric quadratic phase mask along one dimension is given by

\[
H(u, \psi) = \begin{cases} 
\left(\frac{1}{2} - |u|\right) \text{sinc}\left[\frac{4u}{\pi} \left(\frac{1}{2} - |u|\right)\right] \exp\left[-j2u(\psi - \alpha)\right] & \text{for } 0 \leq |u| \leq \frac{1}{2} \\
+ \frac{\pi}{8\alpha} \left[ C(b_c(u)) - C(a_c(u)) \right] + j \text{signum}(u) \left[ S(b_c(u)) - S(a_c(u)) \right] & \frac{1}{2} \leq |u| \leq 1 \\
\frac{\pi}{8\alpha} \exp\left[j2u(\psi + \alpha)\right] & \text{otherwise}
\end{cases}
\]

where \( a_c(u) \) and \( b_c(u) \) are as shown in Eq. (14), and \( a_c(u) \) and \( b_c(u) \) are as described in Eq. (25). Since the OTF is described by different equations at different regions along the spatial frequency axis, it is necessary to verify continuity at the boundaries of these regions, namely at \( |u| = \frac{1}{2} \). Eqs. (22) and (32) indicate that \( I_1(u, \psi) = I_3(u, \psi) = 0 \) at \( |u| = \frac{1}{2} \). It therefore suffices to show that \( I_2(u, \psi) = H_f(u, \psi) \) at this spatial frequency value. The only difference in these two equations lies in the arguments of the Fresnel integrals and hence the conditions \( a_c(u) = a_f(u) \) and \( b_c(u) = b_f(u) \) at \( |u| = \frac{1}{2} \) are sufficient to prove equality of \( I_2(u, \psi) \) and \( H_f(u, \psi) \). From Eqs. (14) and (25), it is seen that at \( |u| = \frac{1}{2} \),

\[
a_c(\pm \frac{1}{2}) = a_f(\pm \frac{1}{2}) = \left(\frac{4\alpha}{\pi}\right)^{\frac{1}{2}} \left(\frac{\psi - \frac{1}{2}}{2\alpha}\right); \\
b_c(\pm \frac{1}{2}) = b_f(\pm \frac{1}{2}) = \left(\frac{4\alpha}{\pi}\right)^{\frac{1}{2}} \left(\frac{\psi + \frac{1}{2}}{2\alpha}\right). \tag{36}
\]

The OTF is hence continuous at \( |u| = \frac{1}{2} \). A quick sanity check on Eq. (35) may also be performed to verify three key properties of an OTF namely:

1. \( H(0, 0) = 1 \).
2. \( H(-f_s, -f_s) = H^*(f_s, f_s) \).
3. \( |H(f_s, f_s)| \leq |H(0, 0)| \).

The first property is validated by Eq. (34). Visual inspection of the plot of the OTF in Figure 2 supports the third property of the OTF. The Hermitian symmetry required by the second property is easily verified by inspecting \( H_c(u, \psi) \) and \( H_f(u, \psi) \) in Eq. (35). The triangle and the sinc functions in \( I_2(u, \psi) \) and \( I_3(u, \psi) \) are real and have even symmetry about \( u = 0 \). Hermitian symmetry is then imparted by the terms \( \exp[-j2u(\psi - \alpha)] \) and \( \exp[j2u(\psi + \alpha)] \) in \( I_2(u, \psi) \) and \( I_3(u, \psi) \) respectively. In \( I_2(u, \psi) \), the arguments \( a_c(u) \) and \( b_c(u) \) are also real and even-symmetric about \( u = 0 \). Hermitian symmetry is then supplied by the signum function in the complex exponential term. Since \( I_1(u, \psi) \), \( I_2(u, \psi) \) and \( I_3(u, \psi) \) are all Hermitian symmetric, their sum \( H_c(u, \psi) \) also exhibits the same property. In the tails, \( H_f(u, \psi) \) can similarly be shown to exhibit...
Hermitian symmetry since $a_T(u)$ and $b_T(u)$ are real and even-symmetric about $u = 0$, and the *signum* function in the complex exponential term contributes to the required property.

Figure 2 depicts the intensity point spread function (PSF) and MTF of the OSQ phase mask imager for three different misfocus values. The MTF plots of the OSQ phase mask imager shown in Figure 2 indicates that the height of the MTF is fairly constant within the passband for different values of defocus except when $|\psi| \approx \alpha$. The passband of this imaging system when $|\psi| < \alpha$ is defined as the length along the spatial frequency axis bounded by the zero-crossing points of the function $a_T(u)$ or $b_T(u)$, depending on whether the misfocus is positive or negative respectively. In the special case where the magnitude of misfocus equals the strength of the OSQ phase mask, the MTF is significantly raised compared to its counterparts at other defocus values, albeit with a lower spatial frequency bandwidth. Figure 2 also suggests that increasing the magnitude of misfocus decreases the available spatial frequency bandwidth. However, in order to confirm that the bandwidth variation is monotonic with respect to the magnitude of misfocus over all misfocus values, it is essential to study the ambiguity function (AF) plots for this phase mask system.

Figure 3 depicts the magnitude of the AF for the OSQ phase mask system. The region where imaging is possible takes on a double-diamond shape in the AF magnitude plot, with the raised MTFs marking the boundaries of the operating region. The MTFs at the boundaries of this region display a higher dynamic range than those for other misfocus values within the area where imaging is possible.

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previously researched behavior of this system. The plots of magnitude of the AF seen in Figure 3 demonstrate that raising the magnitude of misfocus reduces the available spatial frequency bandwidth. An expression for this bandwidth as a function of the misfocus parameter is evaluated next.

### 3. Available Bandwidth of the OSQ Phase Mask System

It is seen from Figure 3 that misfocus present in the system has a low-pass filtering effect on its OTF, with increasing magnitudes of misfocus serving to diminish the available spatial frequency bandwidth. Since the filtering effect is low-pass in nature, an analysis of the tails of the OTF is first performed to obtain information about the roll off point of the OTF on the spatial frequency axis.

Examining the expression for the tails of the OTF in Eq. (35) reveals that all the frequency dependent terms in the OTF expression that contribute to its magnitude manifest themselves in the Fresnel integrals. Plots (a) and (b) in Figure 4 demonstrate the impact of defocus on the arguments $a_T(u)$ and $b_T(u)$ of these Fresnel integrals in the tails of the OTF. When $a_T(u)$ and $b_T(u)$ are large and on opposite sides of the horizontal axis, the real and imaginary parts of the Fresnel integral terms each vary about unity. For applications in which the misfocus is negative, plots (c) and (d) show that the available bandwidth on the MTF plots extend up to the zero-crossing points of the function $b_T(u)$ seen in plots (a) and (b). On the other hand, when the misfocus is a positive quantity, the bandwidth is determined by the zero-crossing points of the function $a_T(u)$. The available spatial frequency bandwidth of the OSQ phase mask imager is then given by

$$u_c = \frac{\alpha}{\alpha + |\psi|} ; \quad |\psi| \leq \alpha, \quad \alpha > 0 . \quad (37)$$

As seen in Eq. (37), the expression for the available spatial frequency bandwidth is valid as long as the magnitude of misfocus is less than the strength of the phase mask.

Figure 5 illustrates the relationship between

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**Fig. 4.** Effect of misfocus on the bandwidth of OSQ phase mask systems. The left column represents a system with $\alpha = 70\pi$ and no misfocus ($\psi = 0$). The right column is for the same system ($\alpha = 70\pi$), but with $\psi = -30\pi$. It is seen in (c) and (d) that the location where the MTF drop occurs corresponds to the zero crossing points of $b_T(u)$ as seen in (a) and (b).
available spatial frequency bandwidth and the zero-crossing points of the function \( b_T(|u|) \). Three different values of defocus are shown. The top plot showing the function \( b_T(|u|) \) illustrates the movement of the zero-crossing points towards \( u = 0 \) as the magnitude of \( \psi \) is increased. The other two plots show the corresponding frequency cutoffs from the MTF and AF perspectives.

### 4. Imaging under the Special Case of \(|\psi| = \alpha\)

The plots depicting the arguments of the Fresnel integrals in Figure 5 also show that when \(|\psi| = \alpha\), \( b_C(u) \) is zero. The terms, \( a_T(u) \) and \( b_T(u) \) are on the same side of the horizontal axis with \( b_T(u) = 0 \) occurring at \(|u| = \frac{1}{2}\). Therefore in this special case and beyond (\(|\psi| > \alpha\)), the quantity \( H_T(u, \psi) \) does not contribute to the spatial frequency bandwidth of the imager. Eq. (37) indicates a normalized bandwidth of \( \frac{1}{2} \) when \(|\psi| = \alpha\); that is, the spatial frequency bandwidth of the imager is one-half of its diffraction-limited bandwidth under this condition. It therefore falls upon the central portion of the OTF to provide any available bandwidth in this scenario.

The raising of the MTF when the magnitude of misfocus equals the strength of the OSQ phase mask may be better understood by an examination of the analytical expression for \( H_C(u, \psi) \). The value of misfocus is taken to be a negative quantity and therefore the special case of \( \alpha = -\psi \) is considered, where \( \alpha >> 1 \). The analysis for positive misfocus values entails only minor changes as outlined below. Under such a condition, it is seen from Eq. (33) that the arguments of the \( \text{sinc} \) and complex exponential terms in the constituent expression representing \( I_2(u, \psi) \) reduce to zero since \( \alpha + \psi = 0 \).

The magnitudes of these terms therefore reduce to unity, leaving only the \( \text{triangle} \) function of \((\frac{1}{2} - |u|)\), which is a real-valued quantity that takes on a magnitude of \( \frac{1}{2} \) at the zero-frequency location and zero at \(|u| = \frac{1}{2}\). Meanwhile, \( I_1(u, \psi) \) is a complex-valued term (except at \( u = 0 \), where it is real) whose magnitude is rapidly decaying as a function of \(|u|\) due to the relatively large frequency of its \( \text{sinc} \) term. \( I_1(u, \psi) \) thus contributes little to the magnitude of the MTF except at the zero frequency where its value equals \( \frac{1}{2} \).

When the magnitude of misfocus equals \( \alpha \), Eq. (25) indicates that \( b_C(|u|) = 0 \) along the spatial frequency axis for \( 0 \leq |u| \leq \frac{1}{2} \), as seen in Figure 5. The real and imaginary parts of the contribution of the Fresnel integrals in \( I_2(u, \psi) \) therefore vary about \( \frac{1}{2} \), resulting in a stationary value of \( \frac{1}{2}(\pi/8\alpha)^{1/2} \) for

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**Fig. 5. Available bandwidth of the OSQ phase mask imager.** The figure shows the relationship between the zero-crossing points of the function \( b_T(|u|) \) and the spatial frequency bandwidth of this imager as seen from the AF magnitude plot. The expression for available spatial frequency bandwidth given by Eq. (37) is valid as long as the radial lines fall within the double-diamond pattern seen in the AF plot. The MTF at the edges of the AF magnitude plot are raised and have a normalized bandwidth of \( u_c = \frac{1}{2} \).
this term. It must be noted that this magnitude is one-half the stationary height of the MTF seen inside the passband when $|\psi| < \alpha$. For large values of $\alpha$, this magnitude of $I_2(u, \psi)$ is much smaller than $\frac{1}{2}$. The primary contribution to the elevated dynamic range of the OTF thus comes from $I_3(u, \psi)$ and the shape of the MTF is nearly that of a triangle. In applications where the misfocus is a non-negative quantity, the roles of $I_1(u, \psi)$ and $I_3(u, \psi)$ are reversed, as are those of $a_C(|u|)$ and $b_C(|u|)$ in $I_2(u, \psi)$.

Figure 6 illustrates imaging under the special case for $|\psi| = \alpha$. This particular example is shown for $\psi = -\alpha$. The magnitudes of the three constituent terms of $H_C(u, \psi)$ when the defocus magnitude equals the strength of the OSQ phase mask are shown. It must be noted that the sum of the plotted quantities does not represent the magnitude of $H_C(u, \psi)$, as $|I_1(u, \psi)| + |I_2(u, \psi)| + |I_3(u, \psi)|$ is generally greater than $|I_1(u, \psi) + I_2(u, \psi) + I_3(u, \psi)|$ except when the phases of these three constituent terms are zero. Figure 6 also shows the raised MTF under this special condition, and the corresponding radial line for working value of misfocus on the AF magnitude plot. It may be noted that this radial line runs through the middle of the dark band at the boundaries of the operating region of the imager. In applications where the misfocus is positive, the special case of $\psi = \alpha$ would be represented by a radial line through the middle of the other dark band on the AF magnitude plot with a slope of identical magnitude but opposite sign.

5. Conclusion

An analytical approach to evaluating the frequency response of the odd-symmetric quadratic phase mask provides a mathematical formulation of its OTF and paves the way for the design of wavefront coding imagers that could be useful in various applications. Obtaining a mathematical expression for the OTF of an OSQ phase mask imager leads to the determination of the available spatial frequency bandwidth of this system as a function of its working value of misfocus. Such an analysis also identified the special imaging condition that yielded an enhanced dynamic range. In future work, imaging configurations will be designed that exploit the enhanced dynamic range of this special condition and obtain improved noise characteristics in applications such as form factor enhancement and aberration correction.

The authors gratefully acknowledge the support of the Defense Advanced Research Projects Agency.
(DARPA) through a grant (N00014-05-1-0841) with the Office of Naval Research.

6. References


