Convex combinations of more than two points
We first consider the extension to three points in $\mathbb{R}^n$.

Given $a, b, c \in \mathbb{R}^n$ and $v, w, x \in \mathbb{R}$,

$$y(v, w, x) = va + wb + xc,$$

with $v + w + x = 1$ and $v, w, x \geq 0$

is a (parameterized) convex combination of the given points.
Let us do a little algebra, assuming that $v + w \neq 0$.

$$y(v, w, x) = va + wb + xc$$

$$= (v + w) \left[ \left( \frac{v}{v + w} \right)a + \left( \frac{w}{v + w} \right)b \right] + xc$$

$$= (1 - x) \left[ \left( \frac{v}{v + w} \right)a + \left( \frac{w}{v + w} \right)b \right] + xc$$

Define $b' = \left( \frac{v}{v + w} \right)a + \left( \frac{w}{v + w} \right)b$.

Then $b'$ is a convex combination of $a$ and $b$ and $y(v, w, x)$ is a convex combination of $b'$ and $c$. Why?
So, assuming that the points are distinct, what is
\[ \{ \, va + wb + xc : \, v + w + x = 1 \mbox{ and } 0 \leq v, w, x \leq 1 \, \} , \]
the set of all convex combinations of \( a, b, c \in \mathbb{R}^n \)?

We have that
\[ b' = \left( \frac{v}{v + w} \right) a + \left( \frac{w}{v + w} \right) b \quad \text{(with} \quad v + w \neq 0) \]
and
\[ va + wb + xc = (1 - x)b' + xc \quad \text{(with} \quad x = 1 - v - w) \]
\{ va + wb + xc : v + w + x = 1 \text{ and } 0 \leq v, w, x \leq 1 \} \\
consists of all the points on the edges and inside the triangle with corner points \(a, b,\) and \(c\).
We now consider the extension to many points in \( \mathbb{R}^n \).

Let \( \mathcal{A} = \{a^1, \ldots, a^m\} \) be a given finite set of \( m \) points in \( \mathbb{R}^n \). The set \( \mathcal{A} \) can be used to generate different polyhedral objects in \( \mathbb{R}^n \) by combining its elements using various linear operations. The elements of \( \mathcal{A} \) are the generators of the objects they define.
One of these fundamental objects is the **convex hull** of the points in $\mathcal{A}$ (a *polytope*) defined by

$$H(\mathcal{A}) = \left\{ \sum_{i=1}^{m} \lambda_i a^i : \sum_{i=1}^{m} \lambda_i = 1 \text{ and } \lambda_1, \ldots, \lambda_m \geq 0 \right\}$$

The convex hull consists of all convex combinations of the generators.

In $\mathbb{R}^3$ one can visualize the convex hull of many points as a multi-faceted diamond.
Any set is said to be **convex** if it contains all convex combinations of any finite set of points from that set.

\( H(A) \) is convex since it can be shown that a convex combination of convex combinations of given points is a convex combination of those points.

Example:

\[
\frac{1}{2} \left[ \frac{1}{3}a^1 + \frac{2}{3}a^2 \right] + \frac{1}{2} \left[ \frac{2}{3}a^2 + \frac{1}{3}a^3 \right] = \frac{1}{6}a^1 + \frac{2}{3}a^2 + \frac{1}{6}a^3
\]
The minimum cardinality subset \( \mathcal{F} \subset \mathcal{A} \) which generates \( H(\mathcal{A}) \) is called the **frame** of \( H(\mathcal{A}) \).

A frame is to a convex hull what a *basis* is to a linear combination.

In \( \mathbb{R}^3 \) when visualizing the convex hull of many points as a multi-faceted diamond, the corner points are the generators.

It can be shown that a **point is a member of** \( \mathcal{F} \) **if and only if it cannot be written as a strict convex combination of two distinct points of** \( H(\mathcal{A}) \).
The fundamental tool for determining the frame $\mathcal{F}$ from $\mathcal{A}$ is the generic linear program:

\[(LP)\]

\[z = \min \sum_{j \in J} \lambda_j\]

s.t.

\[\sum_{j \in J} \lambda_j a^j = a^k\]

\[\lambda_j \geq 0 \quad , \quad j \in J\]

where $J \subset \mathcal{A}\backslash a^k$. 
Fundamental Results:

If $\mathcal{F} \subset \{ a^j : j \in J \}$, $a^k \neq 0$, and the linear program (LP) is feasible, then $a^k \in \mathcal{F}$ if and only if at optimality $z > 1$.

If $\mathcal{F} \subset \{ a^j : j \in J \}$, $a^k \neq 0$, and the linear program (LP) is infeasible, then $a^k \in \mathcal{F}$. 
Naive approaches to finding $\mathcal{F}$ based on iterative solution of problems of type (LP) have long been used. Computationally they suffer by starting with large size sets $J$ and only slowly decreasing their sizes.

Dula, Helgason, and Hickman have shown how to more efficiently compute $\mathcal{F}$, in part making use of iterative solution of problems of type (LP) starting with small size sets $J$ and slowly increasing their sizes.
References:
