Given a system of m linear equations with n variables ($n \geq m$), a *basic solution* is found by setting $n - m$ *nonbasic variables* equal to 0 and solving for the remaining m *basic variables*.

Note that the system of m equations with the m basic variables will either have a unique solution or no solution.

Example

$$z - x_1 - 2x_2 = 0 \quad (1)$$
$$x_1 + x_2 + s_1 = 4 \quad (2)$$
$$x_1 - 2x_2 + s_2 = 2 \quad (3)$$
$$-2x_1 + x_2 + s_3 = 2 \quad (4)$$

Soln.	Nonbasic Variables	Basic Variables
1)	$x_1 = x_2 = 0$	$s_1 = 4,\ s_2 = 2,\ s_3 = 2,\ z = 0$
2)	$x_2 = s_1 = 0$	$x_1 = 4,\ s_2 = -2,\ s_3 = 10,\ z = 4$
3)	$s_1 = s_3 = 0$	$x_1 = \frac{2}{3},\ x_2 = \frac{10}{3},\ s_2 = 8,\ z = \frac{22}{3}$

The vector-matrix form of the system of equations (1)-(4):

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

z	x_1	x_2	s_1	s_2	s_3		
1	- 1	- 2	0	0	0	0	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	1	-2	0	1	0	2	(Row 2)
0	-2	1	0	0	1	2	(Row 3)

We may use *Elementary Row Operations* (ero's), to solve a system of linear equations or to move from one basic solution to another.

- Multiply any row of the matrix by a nonzero constant.
- Multiply any row, say i, by a nonzero scalar c and add the result to another row, say j.

  For each column ℓ, replace matrix element $M_{j\ell}$ by $M_{j\ell} + cM_{i\ell}$ where M_{rc} is the element in row r, column c.!

Using ero's to make x_1 a basic variable

Add Row 1 to Row 0

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	1	-2	0	1	0	2	(Row 2)
0	-2	1	0	0	1	2	(Row 3)

Add -1 $\times$ Row 1 to Row 2

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	-2	1	0	0	1	2	(Row 3)

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	-2	1	0	0	1	2	(Row 3)

Add 2 $\times$ Row 1 to Row 3

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	3	2	0	1	10	(Row 3)

$$
\begin{aligned}
z - 2x_2 + s_1 &= 4 \\
x_1 + x_2 + s_1 &= 4 \\
-3x_2 - s_1 + s_2 &= -2 \\
3x_2 + 2s_1 + s_3 &= 10
\end{aligned}
$$

Non-basic Variables: x_2 and s_1

Basic Variables: z, x_1, s_2, and s_3

Basic Solution: $z = 4$, $x_1 = 4$, $s_2 = -2$, and $s_3 = 10$

Original matrix:

z	x_1	x_2	s_1	s_2	s_3		
1	- 1	- 2	0	0	0	0	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	1	-2	0	1	0	2	(Row 2)
0	-2	1	0	0	1	2	(Row 3)

Matrix after x_1 becomes basic and s_1 becomes nonbasic:

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	3	2	0	1	10	(Row 3)

To make x_2 basic and s_3 non-basic, *pivot* on Row 3, Column 3.

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	**3**	2	0	1	10	(Row 3)

The pivot operation has two phases: first, we divide the pivot row by the pivot element. In this case, divide Row 3 by 3. In the second phase, we eliminate the variable in the pivot column from all rows except the pivot row. In this case, we want to "zero-out" the x_2 column (except for Row 3).

Divide Row 3 by 3.

z	x_1	x_2	s_1	s_2	s_3		
1	0	- 1	1	0	0	4	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

Eliminate x_2 from Row 0. Add Row 3 to Row 1.

z	x_1	x_2	s_1	s_2	s_3		
1	0	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{22}{3}$	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

z	x_1	x_2	s_1	s_2	s_3		
1	0	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{22}{3}$	(Row 0)
0	1	1	1	0	0	4	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

Eliminate x_2 from Row 1. Add -1 $\times$ Row 3 to Row 1.

z	x_1	x_2	s_1	s_2	s_3		
1	0	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{22}{3}$	(Row 0)
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

z	x_1	x_2	s_1	s_2	s_3		
1	0	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{22}{3}$	(Row 0)
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	(Row 1)
0	0	-3	-1	1	0	-2	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

Eliminate x_2 from Row 2. Add 3 $\times$ Row 3 to Row 2.

z	x_1	x_2	s_1	s_2	s_3		
1	0	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{22}{3}$	(Row 0)
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	(Row 1)
0	0	0	1	1	1	8	(Row 2)
0	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$	(Row 3)

$$
\begin{aligned}
z + \frac{5s_1}{3} + \frac{s_3}{3} &= \frac{22}{3} \\
x_1 + \frac{s_1}{3} - \frac{s_3}{3} &= \frac{2}{3} \\
s_1 + s_2 + s_3 &= 8 \\
x_2 + \frac{2s_1}{3} + \frac{s_3}{3} &= \frac{10}{3}
\end{aligned}
$$

Non-basic Variables: s_1 and s_3

Basic Variables: z, x_1, x_2, and s_2

Basic Solution: $x_1 = \frac{2}{3}$, $x_2 = \frac{10}{3}$, $s_2 = 8$, $z = \frac{22}{3}$