The Minimum Cost Network Flow Problem

Problem Instance: Given a network $G = (N, A)$, with a cost c_{ij}, upper bound u_{ij}, and lower bound ℓ_{ij} associated with each directed arc (i, j), and supply of, or demand for, $b(i)$ units of some commodity at each node.

Supply Nodes: $b(i) > 0$

Transshipment Nodes: $b(i) = 0$

Demand Nodes: $b(i) < 0$

Linear Programming Formulation: The problem is that of finding a minimum-cost feasible *flow*:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = b(i) \qquad \forall i \in N \\ & \ell_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in A \end{aligned}$$

Note the problem is not feasible unless the supplies and demands are balanced (i.e., $\sum_{i\in N} b(i) = 0$).

The Circulation Problem: A MCNFP where $b(i) = 0$ for all nodes.

The Transportation Problem: A MCNFP where

- A *bipartite* network where $N = N_1 \cup N_2$ $(N_1 \cap N_2 = \emptyset)$
- N_1 are supply nodes and N_2 are demand nodes
- $i \in N_1$ and $j \in N_2$ for all $(i, j) \in A$

The Assignment Problem: A Transportation Problem where

- $|N_1| = |N_2|$
- $b(i) = +1$ for all $i \in N_1$ and $b(j) = -1$ for all $j \in N_2$

The Shortest Path Problem: defined on a network with arc costs, but no capacity limits. The objective is to find a path from node s, the *source*, to node t, the *sink*, that minimizes the sum of the arc costs along the path.

To formulate as MCNFP:

- s is a suppy node with $b(s) = 1$.
- t is a demand node with $b(t) = -1$.
- All other nodes ($N \setminus \{s, t\}$) are transshipment nodes.
- $\ell_{ij} = 0$ and $u_{ij} = 1$ for all arcs.

LP Formulation of the Shortest Path Problem

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{subject to} \quad & \sum_{\{j:(s,j)\in A\}} x_{sj} - \sum_{\{j:(j,s)\in A\}} x_{js} = 1, \\
& \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = 0 \ \forall i \in N \setminus \{s,t\}, \\
& \sum_{\{j:(t,j)\in A\}} x_{tj} - \sum_{\{j:(j,t)\in A\}} x_{jt} = -1, \\
& 0 \le x_{ij} \le 1 \qquad \forall (i,j) \in A.
\end{aligned}
$$

The Maximum Flow Problem: defined on a directed network with capacities, and no costs. In addition two nodes are specified, a source node, s, and sink node, t. The objective is the find the maximum possible flow between the source and sink while satisfying the arc capacities.

LP Formulation of the Maximum Flow Problem

$$
\begin{array}{ll}
\max & v \\
\text{subject to} & \displaystyle\sum_{\{j:(s,j)\in A\}} x_{sj} - \sum_{\{j:(j,s)\in A\}} x_{js} = v, \\
& \displaystyle\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = 0 \ \forall i \in N \setminus \{s,t\}, \\
& \displaystyle\sum_{\{j:(t,j)\in A\}} x_{tj} - \sum_{\{j:(j,t)\in A\}} x_{jt} = -v, \\
& \ell_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in A.
\end{array}
$$

The scalar variable v is referred to as the *value* of the flow vector $x = \{x_{ij}\}$.

The Max Flow Problem Formulated as MCNFP

Convert the problem to an equivalent minimum cost circulation problem as follows:

- Let $c_{ij} = 0$ for all $(i, j) \in A$.
- Let $b(i) = 0$ for all $i \in N$.
- Add an arc from s to t with cost $c_{st} = -1$.

$$
\begin{aligned}
\min \quad & -x_{ts} \\
\text{s.t.} \quad & \sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ji} = 0 \ \forall i \in N, \\
& \ell_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i, j) \in A.
\end{aligned}
$$

The Texas Confectionery Company (TCC) produces three types of candy bars at two different plants.

- Houston plant produces Rice Krunchy and Aggie bars.
- Austin plant produces Aggie Bars and Longhorn Bars.
- There are 160 hours of production time available per month at each plant.

Product Name	Production Cost		Product Time (Minutes)	Demand (Units)
	Houston	Austin		
Rice Krunchy Bar	$0.04	-	30	200
Aggie Bar	$0.05	$0.06	20	280
Longhorn Bar	-	$0.06	15	320

(a) Formulate a MCNFP that TCC can solve to determine how to minimize the cost of meeting the demand for its products. Briefly describe what the elements (nodes, arcs, capacities, etc.) of your network model represent.

Hint: How many hours of production time are required to meet the demand for each type of candy bar?

(b) Suppose that TCC wants to plan two months in advance and believes that it will need 240 Rice Krunchy Bars, 360 Aggie Bars and 480 Longhorn bars at the end of next month. If there is time left over in the current month (after the production for this month's demand is finished), then some of these bars can be manufactured this month and held in inventory until they are needed. Extend your MCNFP from part (a) to minimize TCC's cost for meeting the demand for its products over the next two months.

(c) Candy bars that are stored in inventory must be kept in a special storage facility so that they do not become stale. How would you extend your MCNFP from part (b) to account for inventory holding costs of \$0.02 for each Rice Krunchy Bar, \$0.01 for each Aggie bar and \$0.05 for each Longhorn bar?